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FOREWORD

The International Journal on Optimization and Applications (IJOA) volume 01 - issue 02 is an open access, double blind peer-reviewed online journal aiming at publishing high-quality research in all areas of : Applied mathematics, Engineering science, Artificial intelligence, Numerical Methods, Embedded Systems, Electric, Electronic en-gineering, Telecommunication Engineering... the IJOA begins its publication from 2021. This journal is enriched by very important special manuscripts that deal with problems using the latest methods of optimization. It aims to develop new ideas and col-laborations, to be aware of the latest search trends in the optimi-zation techniques and their applications in the various fields..

Finally, I would like to thank all participants who have contributed to the achievement of this journal and in particular the authors who have greatly enriched it with their performing articles.

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Existence Result for a General Nonlinear Degenerate Elliptic Problems with Measure Datum in Weighted Sobolev Spaces

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***Abstract*—In this paper, we study the Dirichlet problem associated to the degenerate nonlinear elliptic equations**

$$\begin{cases} Lu(x) = \mu & \textbf{in } \Omega, \\ u(x) = 0 & \textbf{on } \partial\Omega, \end{cases}$$

where

$$Lu(x) = -\mathrm{div}\Big[\omega_1(x)\mathcal{A}(x, \nabla u(x)) + \omega_2(x)\mathcal{B}(x, u(x), \nabla u(x))\Big] + \omega_1(x)g(x, u(x)) + \omega_2(x)\mathcal{H}(x, u(x), \nabla u(x)),$$

is a second order degenerate elliptic operator, with $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are Caratéodory functions, who satisfies some conditions, and the right-hand side term μ belongs to $L^1(\Omega) + \prod_{j=1}^{n} L^{p'}(\Omega, \omega_1^{1-p'})$, ω_1 and ω_2 are weight functions that will be defined in the preliminaries.

***Index Terms*—Nonlinear degenerate elliptic equations, Dirichlet problem, weighted Sobolev spaces, weak solution**

I. Introduction

Let Ω be a bounded open subset in $\mathbb{R}^n$ ($n \geq 2$), $\partial\Omega$ its boundary and $p > 1$ and ω_1, ω_2 are two weights functions in Ω (ω_1 and ω_2 are measurable and strictly positive a.e. in Ω). Let us consider the following nonlinear degenerate elliptic problem

$$\begin{cases} Lu(x) = \mu & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where, L is a second order degenerate elliptic operator

$$Lu(x) = -\mathrm{div}\Big[\omega_1(x)\mathcal{A}(x, \nabla u(x)) + \omega_2(x)\mathcal{B}(x, u(x), \nabla u(x))\Big] + \omega_1(x)g(x, u(x)) + \omega_2(x)\mathcal{H}(x, u(x), \nabla u(x)), \tag{2}$$

and

$$\mu = f_0 - \sum_{j=1}^{n} D_j f_j, \tag{3}$$

with $f_0 \in L^1(\Omega)$ and for $j = 1, ..., n$, $f_j \in L^{p'}(\Omega, \omega_1^{1-p'})$. Furthermore, the functions $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are Caratéodory functions, who satisfying the assumptions of growth, ellipticity and monotonicity.

In the past decade, much attention has been devoted to nonlinear elliptic equations because of their wide application to physical models such as non-Newtonian fluids, boundary layer phenomena for viscous fluids, and chemical heterogenous model, we mention some works in this direction [1], [4], [5], [7]. One of the motivations for studying (1) comes from applications to electrorheological fluids (see [19] for more details) as an important class of non-Newtonian fluids.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial difierential equations. For degenerate partial differential equations, where we have equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces [2], [4], [12], [13], [16]. The type of a weight depends on the equation type.

For $\omega_1 \equiv \omega_2 \equiv 1$ (the non weighted case) and $\mathcal{A}(x, \nabla u) \equiv g \equiv 0$, Equation of the from (1) have been widely studied in [10], where the authors obtain some existence results for the solutions (see also the references therein).

Boccardo et al. [6] considered the nonlinear boundary value problem

$$-\mathrm{div}(a(x, u, \nabla u)) + g(x, u, \nabla u) = \mu,$$

where $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega)$ and $g(x, u, \nabla u) \in L^1(\Omega)$. By combining the truncation technique with some delicate test functions, the authors showed that the problem has a solution $u \in W_0^{1,p}(\Omega)$. Furthermore the degenerate case with difierent conditions haven been studied by many authors (we refer to [11], [22] for more details).

In [3], the authors proved the existence results, in the setting of weighted Sobolev spaces, for quasilinear degenerated elliptic problems associated with the following equation $-\text{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = f - \text{div}F$, where g satisfies the sign condition.

In [8] the author proved the existence of solutions for the problem (1) when $\omega_1 \equiv \omega_2$ and $\mathcal{A}(x,\nabla u) \equiv g \equiv 0$. When $\mathcal{H}(x,u,\nabla u) \equiv g \equiv 0$ existence result for the Problem (1) have been shown in [9].

Our objectif, in this paper, is to study equation (1) by adopting Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega_1)$ (see Definition 2.3). By apply the main theorem on monotone operators (see Theorem 2.3), we show that the Problem (1) admits one and only solution $u \in W_0^{1,p}(\Omega,\omega_1)$.

The paper is organized as follows. In Section 2, we give some preliminaries and the definition of weighted Sobolev spaces and some technical lemmas needed in our peper. In Section 3, we make precise all the assumptions on $\mathcal{A}$, $\mathcal{B}$, g, $\mathcal{H}$ and we introduce the notion of weak solution for the Problem (1). Our main result and his proof, the existence and uniqueness of solution to Equation (1), are collected in Section 4. Section 5 is devoted to an example which illustrates our main result.

II. Preliminaries

In this section, we present some definitions, and preliminaries facts which are used throughout this paper.

By a weight, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus,

$$\omega(E) = \int_E \omega(x)dx \qquad \text{for measurable subset} \quad E \subset \mathbb{R}^n.$$

For $0 < p < \infty$, we denote by $L^p(\Omega,\omega)$ the space of measurable functions f on Ω such that

$$||f||_{L^p(\Omega,\omega)} = \left(\int_E |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty,$$

where ω is a weight, and Ω be open in $\mathbb{R}^n$.

It is a well-known fact that the space $L^p(\Omega,\omega)$, endowed with this norm is a Banach space. We also have that the dual space of $L^p(\Omega,\omega)$ is the space $L^{p'}(\Omega,\omega^{1-p'})$.

We now determine conditions on the weight ω that guarantee that functions in $L^p(\Omega,\omega)$ are locally integrable on Ω.

Proposition 2.1: [17], [18] Let $1 \le p < \infty$. If the weight ω is such that

$$\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \qquad \text{if} \quad p > 1,$$

$$ess \sup_{x\in B} \frac{1}{\omega(x)} < +\infty \qquad \text{if} \quad p = 1,$$

for every ball $B \subset \Omega$. Then,

$$L^p(\Omega,\omega) \subset L^1_{loc}(\Omega).$$

As a consequence, under conditions of Proposition 2.1, the convergence in $L^p(\Omega,\omega)$ implies convergence in $L^1_{loc}(\Omega)$. Moreover, every function in $L^p(\Omega,\omega)$ has a distributional derivatives. It thus makes sense to talk about distributional derivatives of functions in $L^p(\Omega,\omega)$.

A class of weights, which is particularly well understood, is the class of A_p-weight that was introduced by B. Muckenhoupt.

Definition 2.1: Let $1 \le p < \infty$. A weight ω is said to be an A_p-weight, or ω belongs to the Muckenhoupt class, if there exists a positive constant $C = C(p,\omega)$ such that, for every ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B (\omega(x))^{\frac{-1}{p-1}}dx\right)^{p-1} \le C \text{ if } p > 1,$$

$$\left(\frac{1}{|B|}\int_B \omega(x)dx\right) ess \sup_{x\in B}\frac{1}{\omega(x)} \le C \quad \text{if} \quad p = 1,$$

where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. The infimum over all such constants C is called the A_p constant of ω. We denote by A_p, $1 \le p < \infty$, the set of all A_p weights.

If $1 \le q \le p < \infty$, then $A_1 \subset A_q \subset A_p$ and the A_q constant of ω equals the A_p constant of ω (we refer to [15], [16], [20] for more informations about A_p-weights).

Example 2.1: (Example of A_p-weights)

(i) If ω is a weight and there exist two positive constants C and D such that $C \le \omega(x) \le D$ for a.e. $x \in \mathbb{R}^n$, then $\omega \in A_p$ for $1 \le p < \infty$.

(ii) Suppose that $\omega(x) = |x|^\eta$, $x \in \mathbb{R}^n$. Then $\omega \in A_p$ if and only if $-n < \eta < n(p-1)$ for $1 \le p < \infty$ (see Corollary 4.4, Chapter IX in [20]).

(iii) Let Ω be an open subset of $\mathbb{R}^n$. Then $\omega(x) = e^{\lambda\varphi(x)} \in A_2$, with $\varphi \in W^{1,n}(\Omega)$ and λ is sufficiently small (see Corollary 2.18 in [15]).

Definition 2.2: A weight ω is said to be doubling, if there exists a positive constant C such that

$$\omega(2B) \le C\omega(B),$$

for every ball $B = B(x,r) \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as B which is twice as large. The infimum over all constants C is called the doubling constant of ω.

It follows directly from the A_p condition and Hölder inequality that an A_p-weight has the following strong doubling property. In particular, every A_p-weight is doubling (see Corollary 15.7 in [16]).

Proposition 2.2: [21] Let $\omega \in A_p$ with $1 \le p < \infty$ and let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then

$$\left(\frac{|E|}{|B|}\right)^p \le C\frac{\omega(E)}{\omega(B)}$$

where C is the A_p constant of ω.

Remark 2.1: If $\omega(E) = 0$ then $|E| = 0$. The measure ω and the Lebesgue measure $|.|$ are mutually absolutely continuous,

that is they have the same zero sets $\big(\omega(E) = 0$ if and only if $|E| = 0\big)$; so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..
The weighted Sobolev space $W^{1,p}(\Omega,\omega)$ is defined as follows.

Definition 2.3: Let $\Omega \subset \mathbb{R}^n$ be open, and let ω be an A_p-weight, $1 \leq p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D_j u \in L^p(\Omega,\omega)$, for $j = 1,...,n$. The norm of u in $W^{1,p}(\Omega,\omega)$ is given by

$$||u||^p_{W^{1,p}(\Omega,\omega)} = \int_\Omega |u(x)|^p\omega(x)dx + \sum_{j=1}^{n}\int_\Omega |D_j u(x)|^p\omega(x)dx.$$

We also define $W^{1,p}_0(\Omega,\omega)$ as the closure of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}(\Omega,\omega)$ with respect to the norm $||.||_{W^{1,p}(\Omega,\omega)}$. Note that $C^\infty_0(\Omega)$ is dense in $W^{1,p}_0(\Omega,\omega)$.
Equipped by this norm, $W^{1,p}(\Omega,\omega)$ and $W^{1,p}_0(\Omega,\omega)$ are separable and reflevixe Banach spaces (see Proposition 2.1.2. in [21] and see [18] for more informations about the spaces $W^{1,p}(\Omega,\omega)\big)$. The dual of space $W^{1,p}_0(\Omega,\omega)$ is the space $W^{-1,p'}_0(\Omega,\omega^{1-p'})$.

Let us give the following theorems which are needed later.

Theorem 2.1: [14] Let $\omega \in A_p$, $1 \leq p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \longrightarrow u$ in $L^p(\Omega,\omega)$, then there exist a subsequence (u_{m_k}) and a function $\Phi \in L^p(\Omega,\omega)$ such that

(i) $u_{m_k}(x) \longrightarrow u(x)$, $m_k \longrightarrow \infty$, ω-a.e. on Ω.
(ii) $|u_{m_k}(x)| \leq \Phi(x)$, ω-a.e. on Ω.

Theorem 2.2: [11] (The weighted Sobolev inequality) Let $\omega \in A_p$, $1 \leq p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. There exist constants C_Ω and δ positive such that for all $u \in W^{1,p}_0(\Omega,\omega)$ and all θ satisfying $1 \leq \theta \leq \frac{n}{n-1} + \delta$,

$$||u||_{L^{\theta p}(\Omega,\omega)} \leq C_\Omega ||\nabla u||_{L^p(\Omega,\omega)},$$

where C_Ω depends only on n, p, the A_p constant of ω and the diameter of Ω.

Theorem 2.3: [22] Let $A : X \longrightarrow X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:

(a) For each $T \in X^*$, the equation $Au = T$ has a solution $u \in X$.
(b) If the operator A is strictly monotone, then equation $Au = T$ has a unique solution $u \in X$.

III. Basic assumptions and notion of solutions

A. Basic assumptions

Let us now give the precise hypotheses on the Problem (1), we assume that the following assumptions: Ω be a bounded open subset of $\mathbb{R}^n$($n \geq 2$), $1 < q < p < \infty$, let ω_1 and ω_2 are two weights functions, and let $\mathcal{A}_j : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathcal{B}_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ $(j = 1,...,n)$, with $\mathcal{B}(x,\eta,\xi) = \Big(\mathcal{B}_1(x,\eta,\xi),...,\mathcal{B}_n(x,\eta,\xi)\Big)$ and $\mathcal{A}(x,\xi) = \Big(\mathcal{A}_1(x,\xi),...,\mathcal{A}_n(x,\xi)\Big)$, $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the following assumptions:

(A1) For $j = 1,...,n$, $\mathcal{B}_j$, $\mathcal{A}_j$, g and $\mathcal{H}$ are Caratéodory functions.

(A2) There are positive functions $h_1,\ h_2,\ h_3,\ h_4,\ h_5,\ h_6 \in L^\infty(\Omega)$ and $K_1,\ K_4 \in L^{p'}(\Omega,\omega_1)\Big($with $\frac{1}{p} + \frac{1}{p'} = 1\Big)$ and $K_2,\ K_3 \in L^{q'}(\Omega,\omega_2)\Big($with $\frac{1}{q} + \frac{1}{q'} = 1\Big)$ such that :

$$|\mathcal{A}(x,\xi)| \leq K_1(x) + h_1(x)|\xi|^{\frac{p}{p'}},$$

$$|\mathcal{B}(x,\eta,\xi)| \leq K_2(x) + h_2(x)|\eta|^{\frac{q}{q'}} + h_3(x)|\xi|^{\frac{q}{q'}},$$

$$|g(x,\eta)| \leq K_4(x) + h_6(x)|\eta|^{\frac{p}{p'}},$$

and

$$|\mathcal{H}(x,\eta,\xi)| \leq K_3(x) + h_4(x)|\eta|^{\frac{q}{q'}} + h_5(x)|\xi|^{\frac{q}{q'}}.$$

(A3) There exists a constant $\alpha > 0$ such that :

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\xi'),\xi - \xi'\rangle \geq \alpha|\xi - \xi'|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'),\xi - \xi'\rangle \geq 0,$$

$$\Big(g(x,\eta) - g(x,\eta')\Big)\Big(\eta - \eta'\Big) \geq 0,$$

and

$$\Big(\mathcal{H}(x,\eta,\xi) - \mathcal{H}(x,\eta',\xi')\Big)\Big(\eta - \eta'\Big) \geq 0,$$

whenever $(\eta,\xi),(\eta',\xi') \in \mathbb{R} \times \mathbb{R}^n$ with $\eta \neq \eta'$ and $\xi \neq \xi'$ $\Big($where $\langle .,.\rangle$ denotes here the usual inner product in $\mathbb{R}^n\Big)$.

(A4) There are constants $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 > 0$ such that :

$$\langle \mathcal{A}(x,\xi),\xi\rangle \geq \lambda_1|\xi|^p,$$

$$\langle \mathcal{B}(x,\eta,\xi),\xi\rangle \geq \lambda_2|\xi|^q + \lambda_3|\eta|^q,$$

$$g(x,\eta)\eta \geq \lambda_4|\eta|^p,$$

and

$$\mathcal{H}(x,\eta,\xi)\eta \geq 0.$$

B. Notions of solutions

The definition of a weak solution for Problem (1) can be stated as follows.

Definition 3.1: We say that an element $u \in W^{1,p}_0(\Omega,\omega_1)$ is a weak solution of Problem (1) if :

$$\int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla\varphi\rangle\omega_1 dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle\omega_2 dx + \int_\Omega g(x,u)\varphi\omega_1 dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\varphi\omega_2 dx = \int_\Omega f_0\varphi dx + \sum_{j=1}^{n}\int_\Omega f_j D_j\varphi dx,$$

for all $\varphi \in W^{1,p}_0(\Omega,\omega_1)$.

Remark 3.1: We seek to establish a relationship between ω_1 and ω_2, in order to ensure the existence and uniqueness of solution for our Problem (1). At first we notice if $\frac{\omega_2}{\omega_1} \in$

$L^s(\Omega,\omega_1)$ where $s=\frac{p}{p-q}$, $1<q<p<\infty$ and $\omega_1,\omega_2\in A_p$, then, by Hölder inequality we obtain

$$||u||_{L^q(\Omega,\omega_2)}\leq C_{p,q}||u||_{L^p(\Omega,\omega_1)},$$

where $C_{p,q}=||\frac{\omega_2}{\omega_1}||^{1/q}_{L^s(\Omega,\omega_1)}$.

IV. Main result

A. Result on the existence and uniqueness

In this subsection we will state the existence and uniqueness of solution to Problem (1) in Theorem 4.1. In the next subsections we will present the proof.

Theorem 4.1: Let $1<q<p<\infty$ and assume that **(A1)** – **(A4)** holds. If

- (i) $f_0/\omega_2\in L^{q'}(\Omega,\omega_2)$ and $f_j/\omega_1\in L^{p'}(\Omega,\omega_1)$ $(j=1,...,n)$..
- (ii) $\omega_1,\omega_2\in A_p$ such that $\frac{\omega_2}{\omega_1}\in L^s(\Omega,\omega_1)$, where $s=\frac{p}{p-q}$.

Then, the Problem (1) has only one solution $u\in W^{1,p}_0(\Omega,\omega_1)$.

B. Proof of Theorem 4.1

The basic idea of our proof is to reduce the Problem (1) to an operator equation $\mathbf{A}u=\mathbf{T}$ and apply the Theorem 2.3. The proof of Theorem 4.1 will be divided into several steps.

1) Equivalent operator equation: In this subsection, we use the somme tools and the condition **(A2)** to prove an existence the operator $\mathbf{A}$ such that the Problem (1) is equivalent to the operator equation $\mathbf{A}u=\mathbf{T}$. We introduce the operators

$$\begin{aligned}&\mathbf{T}:W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &\varphi\longrightarrow\mathbf{T}(\varphi)=\int_\Omega f_0\varphi dx+\sum_{j=1}^n\int_\Omega f_jD_j\varphi dx,\end{aligned}$$

and

$$\begin{aligned}&\mathbf{B}:W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &(u,\varphi)\longrightarrow\mathbf{B}_1(u,\varphi)+\mathbf{B}_2(u,\varphi)+\mathbf{B}_3(u,\varphi)+\mathbf{B}_4(u,\varphi),\end{aligned}$$

where, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$ and $\mathbf{B}_4$ are defined as follows

$$\begin{aligned}&\mathbf{B}_1:W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &\mathbf{B}_1(u,\varphi)=\int_\Omega\langle\mathcal{A}(x,\nabla u),\nabla\varphi\rangle\omega_1dx,\end{aligned}$$

$$\begin{aligned}&\mathbf{B}_2:W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &\mathbf{B}_2(u,\varphi)=\int_\Omega\langle\mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle\omega_2dx,\end{aligned}$$

$$\begin{aligned}&\mathbf{B}_3:W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &\mathbf{B}_3(u,\varphi)=\int_\Omega g(x,u)\varphi\omega_1dx.\end{aligned}$$

$$\begin{aligned}&\mathbf{B}_4:W^{1,p}_0(\Omega,\omega_1)\times W^{1,p}_0(\Omega,\omega_1)\longrightarrow\mathbb{R}\\ &\mathbf{B}_4(u,\varphi)=\int_\Omega\mathcal{H}(x,u,\nabla u)\varphi\omega_2dx.\end{aligned}$$

Then $u\in W^{1,p}_0(\Omega,\omega_1)$ is a weak solution of Problem (1) if and only if

$$\mathbf{B}(u,\varphi)=\mathbf{T}(\varphi),\qquad\text{for all }\ \varphi\in W^{1,p}_0(\Omega,\omega_1).$$

We will show that $\mathbf{T}\in W^{-1,p'}_0(\Omega,\omega_1^{1-p'})$ and $\mathbf{B}(u,.)$ is linear, for each $u\in W^{1,p}_0(\Omega,\omega_1)$.

(i) Using Hölder inequality and Theorem 2.2(with $\theta=1$), we obtain

$$\begin{aligned}&|\mathbf{T}(\varphi)|\\ &\leq\int_\Omega|f_0|\,|\varphi|\,dx+\sum_{j=1}^n\int_\Omega|f_j|\,|D_j\varphi|\,dx\\ \leq&\left(C_{p,q}||f_0/\omega_2||_{L^{q'}(\Omega,\omega_2)}+\sum_{j=1}^n||f_j/\omega_1||_{L^{p'}(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

According to $f_0/\omega_2\in L^{q'}(\Omega,\omega_2)$ and $f/\omega_1\in L^{p'}(\Omega,\omega_1)$, we deduce that $\mathbf{T}\in W^{-1,p'}_0(\Omega,\omega_1^*)$.

(ii) Let $u\in W^{1,p}_0(\Omega,\omega_1)$. We have

$$|\mathbf{B}(u,\varphi)|\quad\leq\quad|\mathbf{B}_1(u,\varphi)|+|\mathbf{B}_2(u,\varphi)|+|\mathbf{B}_3(u,\varphi)|+|\mathbf{B}_4(u,\varphi)|.\tag{4}$$

In (4), by **(A2)**, Hölder inequality, Remark 3.1 and Theorem 2.2(with $\theta=1$), we have

$$\begin{aligned}|\mathbf{B}_1(u,\varphi)|\quad&\leq\quad\int_\Omega|\mathcal{A}(x,\nabla u)||\nabla\varphi|\omega_1dx\\ &\leq\quad\int_\Omega\left(K_1+h_1|\nabla u|^{\frac{p}{p'}}\right)|\nabla\varphi|\omega_1dx\\ \leq&\left(||K_1||_{L^{p'}(\Omega,\omega_1)}+||h_1||_{L^\infty(\Omega)}||u||^{\frac{p}{p'}}_{W^{1,p}_0(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)},\end{aligned}$$

and

$$\begin{aligned}|\mathbf{B}_2(u,\varphi)|\quad&\leq\quad\int_\Omega|\mathcal{B}(x,u,\nabla u)||\nabla\varphi|\omega_2dx\\ &\leq\quad\int_\Omega\left(K_2+h_2|u|^{\frac{q}{q'}}+h_3|\nabla u|^{\frac{q}{q'}}\right)|\nabla\varphi|\omega_2dx\\ \leq&\,||K_2||_{L^{q'}(\Omega,\omega_2)}C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}+||h_2||_{L^\infty(\Omega)}C^{\frac{q}{q'}}_{p,q}||u||^{\frac{q}{q'}}_{L^p(\Omega,\omega_1)}\\ &C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}+||h_3||_{L^\infty(\Omega)}C^{\frac{q}{q'}}_{p,q}||\nabla u||^{\frac{q}{q'}}_{L^p(\Omega,\omega_1)}C_{p,q}||\nabla\varphi||_{L^p(\Omega,\omega_1)}\\ \leq&\Big[C_{p,q}||K_2||_{L^{q'}(\Omega,\omega_2)}+C^q_{p,q}\left(||h_2||_{L^\infty(\Omega)}+||h_3||_{L^\infty(\Omega)}\right)\\ &||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\Big]\,||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

Analogously, we have

$$\begin{aligned}|\mathbf{B}_3(u,\varphi)|\quad&\leq\quad\int_\Omega|g(x,u)||\varphi|\omega_1dx\\ \leq&\left(||K_4||_{L^{p'}(\Omega,\omega_1)}+||h_6||_{L^\infty(\Omega)}||u||^{\frac{p}{p'}}_{W^{1,p}_0(\Omega,\omega_1)}\right)||\varphi||_{W^{1,p}_0(\Omega,\omega_1)},\end{aligned}$$

and

$$\begin{aligned}|\mathbf{B}_4(u,\varphi)|\quad&\leq\quad\int_\Omega|\mathcal{H}(x,u,\nabla u)||\varphi|\omega_2dx\\ \leq&\Big[C_{p,q}||K_3||_{L^{q'}(\Omega,\omega_2)}+C^q_{p,q}\left(||h_4||_{L^\infty(\Omega)}+||h_5||_{L^\infty(\Omega)}\right)\\ &||u||^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\Big]\,||\varphi||_{W^{1,p}_0(\Omega,\omega_1)}.\end{aligned}$$

Hence, in (4) we obtain, for all $u \in W_0^{1,p}(\Omega,\omega_1)$,

$$\begin{aligned}&|\mathbf{B}(u,\varphi)|\\ &\leq \Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)} + ||K_4||_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\Big(\|K_3\|_{L^{q'}(\Omega,\omega_2)}\\ &+\|K_2\|_{L^{q'}(\Omega,\omega_2)}\Big) + \Big(\|h_1\|_{L^\infty(\Omega)} + \|h_6\|_{L^\infty(\Omega)}\Big)\|u\|^{\frac{p}{p'}}_{W_0^{1,p}(\Omega,\omega_1)}\\ &+C^q_{p,q}\Big(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)} + ||h_4||_{L^\infty(\Omega)} + ||h_5||_{L^\infty(\Omega)}\Big)\\ &\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1)}\Big]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.\end{aligned}$$

Since $\mathbf{B}(u,.)$ is linear and continuous, for each $u \in W_0^{1,p}(\Omega,\omega_1)$, there exists a linear and continuous operator denoted by $\mathbf{A} : W_0^{1,p}(\Omega,\omega_1) \longrightarrow W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$ such that

$$\langle \mathbf{A}u,\varphi\rangle = \mathbf{B}(u,\varphi), \quad \text{for all } u,\varphi \in W_0^{1,p}(\Omega,\omega_1),$$

where $\langle f,x\rangle$ denotes the value of the linear functional f at the point x . Moreover, we have

$$\begin{aligned}&\|\mathbf{A}u\|_*\\ &\leq \|K_1\|_{L^{p'}(\Omega,\omega_1)} + ||K_4||_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\Big(\|K_3\|_{L^{q'}(\Omega,\omega_2)}\\ &+\|K_2\|_{L^{q'}(\Omega,\omega_2)}\Big) + \Big(\|h_1\|_{L^\infty(\Omega)} + \|h_6\|_{L^\infty(\Omega)}\Big)\|u\|^{\frac{p}{p'}}_{W_0^{1,p}(\Omega,\omega_1)}\\ &+C^q_{p,q}\Big(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)} + ||h_4||_{L^\infty(\Omega)} + ||h_5||_{L^\infty(\Omega)}\Big)\\ &\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1)},\end{aligned}$$

where

$$\|\mathbf{A}u\|_* = \sup\Big\{|\langle \mathbf{A}u,\varphi\rangle| = |\mathbf{B}(u,\varphi)| : \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} = 1\Big\}$$

is the norm in $W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$.

Consequently, Problem (1) is equivalent to the operator equation

$$\mathbf{A}u = \mathbf{T},\ u \in W_0^{1,p}(\Omega,\omega_1).$$

2) Coercivity of the operator $\mathbf{A}$*:* In this step, we prove that the operator $\mathbf{A}$ is coercive. To this purpose let $u \in W_0^{1,p}(\Omega,\omega_1)$, we have

$$\begin{aligned}&\langle \mathbf{A}u,u\rangle = \mathbf{B}(u,u)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla u\rangle\omega_1 dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla u\rangle \omega_2 dx\\ &\quad + \int_\Omega g(x,u)u\omega_1 dx + \int_\Omega \mathcal{H}(x,u,\nabla u)u\omega_2 dx.\end{aligned}$$

Moreover, from **(A4)** and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\langle \mathbf{A}u,u\rangle &\geq \lambda_1\int_\Omega |\nabla u|^p\omega_1 dx + \lambda_2\int_\Omega |\nabla u|^q\omega_2 dx\\ &\quad + \lambda_3\int_\Omega |u|^q\omega_2 dx + \lambda_4\int_\Omega |u|^p\omega_1 dx\\ &\geq min(\lambda_1,\lambda_4)\Big[\int_\Omega |\nabla u|^p\omega_1 dx + \int_\Omega |u|^p\omega_1 dx\Big]\\ &\quad + min(\lambda_2,\lambda_3)\Big[\int_\Omega |\nabla u|^q\omega_2 dx + \int_\Omega |u|^q\omega_2 dx\Big]\\ &= min(\lambda_1,\lambda_4)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} + min(\lambda_2,\lambda_3)\|u\|^q_{W_0^{1,q}(\Omega,\omega_2)}\\ &\geq min(\lambda_1,\lambda_4)\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}.\end{aligned}$$

Hence, we obtain

$$\frac{\langle \mathbf{A}u,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \geq min(\lambda_1,\lambda_4)\|u\|^{p-1}_{W_0^{1,p}(\Omega,\omega_1)}.$$

Therefore, since $p > 1$, we have

$$\frac{\langle \mathbf{A}u,u\rangle}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \longrightarrow +\infty \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \longrightarrow +\infty,$$

that is, $\mathbf{A}$ is coercive.

3) Monotonicity of the operator $\mathbf{A}$*:* The operator $\mathbf{A}$ is strictly monotone. In fact, for all $u_1,u_2 \in W_0^{1,p}(\Omega,\omega_1)$ with $u_1 \neq u_2$, we have

$$\begin{aligned}&\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1 - u_2\rangle = \mathbf{B}(u_1,u_1-u_2) - \mathbf{B}(u_2,u_1-u_2)\\ &= \int_\Omega \langle \mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2),\nabla(u_1-u_2)\rangle\omega_1 dx\\ &+ \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1) - \mathcal{B}(x,u_2,\nabla u_2),\nabla(u_1-u_2)\rangle\omega_2 dx\\ &+ \int_\Omega \Big(g(x,u_1) - g(x,u_2)\Big)\Big(u_1-u_2\Big)\omega_1 dx\\ &+ \int_\Omega \Big(\mathcal{H}(x,u_1,\nabla u_1) - \mathcal{H}(x,u_2,\nabla u_2)\Big)\Big(u_1-u_2\Big)\omega_2 dx.\end{aligned}$$

Thanks to **(A3)**, we obtain

$$\begin{aligned}\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1-u_2\rangle &\geq \int_\Omega \alpha|\nabla(u_1-u_2)|^p\omega_1 dx\\ &\geq \alpha\|\nabla(u_1-u_2)\|^p_{L^p(\Omega,\omega_1)},\end{aligned}$$

and by Theorem 2.2(with $\theta = 1$), we conclude that

$$\langle \mathbf{A}u_1 - \mathbf{A}u_2, u_1-u_2\rangle \geq \frac{\alpha}{(C^p_\Omega+1)}\|u_1-u_2\|^p_{W_0^{1,p}(\Omega,\omega_1)}.$$

Therefore, the operator $\mathbf{A}$ is strictly monotone.

4) Continuity of the operator $\mathbf{A}$*:* We need to show that the operator $\mathbf{A}$ is continuous. To this purpose let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. Note that if $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$, then $u_m \longrightarrow u$ in $L^p(\Omega,\omega_1)$ et $\nabla u_m \longrightarrow \nabla u$ in $(L^p(\Omega,\omega_1))^n$. Hence, thanks to Theorem 2.1, there exist a subsequence (u_{m_k}), functions $\Phi_1 \in L^p(\Omega,\omega_1)$ and $\Phi_2 \in L^p(\Omega,\omega_1)$ such that

$$\begin{aligned}&u_{m_k}(x) \longrightarrow u(x), && \omega_1 - a.e. \text{ in } \Omega\\ &|u_{m_k}(x)| \leq \Phi_1(x), && \omega_1 - a.e. \text{ in } \Omega\\ &\nabla u_{m_k}(x) \longrightarrow \nabla u(x), && \omega_1 - a.e. \text{ in } \Omega\\ &|\nabla u_{m_k}(x)| \leq \Phi_2(x), && \omega_1 - a.e. \text{ in } \Omega.\end{aligned} \tag{5}$$

We will show that $\mathbf{A}u_m \longrightarrow \mathbf{A}u$ in $W_0^{-1,p'}(\Omega,\omega_1^{1-p'})$. In order to prove this convergence we proceed in four steps.

Step 1:

For $j = 1,...,n$, we define the operator

$$\begin{aligned}&F_j : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{p'}(\Omega,\omega_1)\\ &(F_j u)(x) = \mathcal{A}_j(x,\nabla u(x)).\end{aligned}$$

We now show that the operator F_j is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)** and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |\mathcal{A}_j(x,\nabla u)|^{p'}\omega_1 dx\\ &\le \int_\Omega \left(K_1 + h_1|\nabla u|^{\frac{p}{p'}}\right)^{p'}\omega_1 dx\\ &\le C_p \int_\Omega \left(K_1^{p'} + h_1^{p'}|\nabla u|^p\right)\omega_1 dx\\ &\le C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\nabla u\|^p_{L^p(\Omega,\omega_1)}\right]\\ &\le C_p\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right],\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $F_j u_m \longrightarrow F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue's theorem and the convergence principle in Banach spaces.
By **(A2)**, we obtain

$$\begin{aligned}&\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)}\\ &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega_1 dx\\ &\le \int_\Omega (|\mathcal{A}_j(x,\nabla u_{m_k})| + |\mathcal{A}_j(x,\nabla u)|)^{p'}\omega_1 dx\\ &\le C_p\int_\Omega \left(|\mathcal{A}_j(x,\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,\nabla u)|^{p'}\right)\omega_1 dx\\ &\le C_p\int_\Omega \left[\left(K_1 + h_1|\nabla u_{m_k}|^{\frac{p}{p'}}\right)^{p'} + \left(K_1 + h_1|\nabla u|^{\frac{p}{p'}}\right)^{p'}\right]\omega_1 dx\\ &\le 2C_pC_p'\int_\Omega\left(K_1^{p'} + h_1^{p'}\Phi_2^p\right)\omega_1 dx\\ &\le 2C_pC_p'\left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\Phi_2\|^p_{L^p(\Omega,\omega_1)}\right].\end{aligned}$$

Hence, thanks to **(A1)**, we get, as $k \longrightarrow \infty$

$$F_j u_{m_k}(x) = \mathcal{A}_j(x,\nabla u_{m_k}(x)) \longrightarrow \mathcal{A}_j(x,\nabla u(x)) = F_j u(x),$$

for almost all $x \in \Omega$. Therefore, by Lebesgue's theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$F_j u_{m_k} \longrightarrow F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

Finally, in view to convergence principle in Banach spaces, we have

$$F_j u_m \longrightarrow F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1). \tag{6}$$

Step 2:
For $j = 1,...,n$, we define the operator

$$\begin{aligned}G_j &: W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{q'}(\Omega,\omega_2)\\ (G_j u)(x) &= \mathcal{B}_j(x,u(x),\nabla u(x)).\end{aligned}$$

We also have that the operator G_j is continuous and bounded. In fact,

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)**, Remark 3.1 and Theorem 2.2(with $\theta = 1$), we obtain

$$\begin{aligned}\|G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |\mathcal{B}_j(x,u,\nabla u)|^{q'}\omega_2 dx\\ &\le \int_\Omega\left(K_2 + h_2|u|^{\frac{q}{q'}} + h_3|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\\ &\le C_q\int_\Omega\left[K_2^{q'} + h_2^{q'}|u|^q + h_3^{q'}|\nabla u|^q\right]\omega_2 dx\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\omega_2)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^q(\Omega,\omega_2)}\Big]\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla u\|^q_{L^p(\Omega,\omega_1)}\Big]\\ &\le C_q\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\Big(\|h_2\|^{q'}_{L^\infty(\Omega)}\\ &\quad + \|h_3\|^{q'}_{L^\infty(\Omega)}\Big)\|u\|^q_{W_0^{1,p}(\Omega,\omega_1)}\Big],\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We will show that $G_j u_m \longrightarrow G_j u$ in $L^{q'}(\Omega,\omega_2)$.
According to **(A2)** and Remark 3.1, we obtain

$$\begin{aligned}&\|G_j u_{m_k} - G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)} = \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{q'}\omega_2 dx\\ &\le \int_\Omega\Big(|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k}| + |\mathcal{B}_j(x,u,\nabla u)|\Big)^{q'}\omega_2 dx\\ &\le C_q\int_\Omega\Big(|\mathcal{B}_j(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x,u,\nabla u)|^{q'}\Big)\omega_2 dx\\ &\le C_q\Big[\int_\Omega\Big(K_2 + h_2|u_{m_k}|^{\frac{q}{q'}} + h_3|\nabla u_{m_k}|^{\frac{q}{q'}}\Big)^{q'}\omega_2 dx\\ &\qquad + \int_\Omega\Big(K_2 + h_2|u|^{\frac{q}{q'}} + h_3|\nabla u|^{\frac{q}{q'}}\Big)^{q'}\omega_2 dx\Big]\\ &\le 2C_qC_q'\int_\Omega\Big(K_2^{q'} + h_2^{q'}\Phi_1^q + h_3^{q'}\Phi_2^q\Big)\omega_2 dx\\ &\le 2C_qC_q'\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_2\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^q(\Omega,\omega_2)}\\ &\qquad + \|h_3\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^q(\Omega,\omega_2)}\Big]\\ &\le 2C_qC_q'\Big[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\|h_2\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^p(\Omega,\omega_1)}\\ &\qquad + C^q_{p,q}\|h_3\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^p(\Omega,\omega_1)}\Big].\end{aligned}$$

Then, by **(A1)**, we have, as $k \longrightarrow \infty$

$$G_j u_{m_k}(x) \longrightarrow G_j u(x), \qquad \text{a.e. } x \in \Omega.$$

Therefore, in view to Lebesgue's theorem, we have

$$\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$G_j u_{m_k} \longrightarrow G_j u \quad \text{in} \quad L^{q'}(\Omega,\omega_2).$$

Hence, from the convergence principle in Banach spaces, we conclude that

$$G_j u_m \longrightarrow G_j u \quad \text{in} \quad L^{q'}(\Omega,\omega_2). \tag{7}$$

Step 3:
We define the operator

$$H : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{p'}(\Omega,\omega_1)$$
$$(Hu)(x) = g(x, u(x)).$$

In this step, we will show that the operator H is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)**, we obtain

$$\begin{aligned}
\|Hu\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |g(x,u)|^{p'}\omega_1 dx\\
&\le \int_\Omega \left(K_4 + h_6|u|^{\frac{p}{p'}}\right)^{p'} \omega_1 dx\\
&\le C_p \int_\Omega \left(K_4^{p'} + h_6^{p'}|u|^p\right)\omega_1 dx\\
&\le C_p \left[\|K_4\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{L^p(\Omega,\omega_1)}\right]\\
&\le C_p \left[\|K_3\|_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right],
\end{aligned}$$

where the constant C_p depends only on p.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $Hu_m \longrightarrow Hu$ in $L^{p'}(\Omega,\omega_1)$.
By **(A2)** , we get

$$\begin{aligned}
&\|Hu_{m_k} - Hu\|^{p'}_{L^{p'}(\Omega,\omega_1)}\\
&= \int_\Omega |Hu_{m_k}(x) - hu(x)|^{p'}\omega_1 dx\\
&\le \int_\Omega \left(|g(x,u_{m_k})| + |g(x,u)|\right)^{p'} \omega_1 dx\\
&\le C_p \int_\Omega \left(|g(x,u_{m_k})|^{p'} + |g(x,u)|^{p'}\right)\omega_1 dx\\
&\le C_p \int_\Omega \left[(K_4 + h_6|u_{m_k}|^{\frac{p}{p'}})^{p'} + (K_4 + h_6|u|^{\frac{p}{p'}})^{p'}\right]\omega_1 dx\\
&\le 2C_pC'_p \int_\Omega \left(K_4^{p'} + h_6^{p'}\Phi_1^p\right)\omega_1 dx\\
&\le 2C_pC'_p \left[\|K_4\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_6\|^{p'}_{L^\infty(\Omega)}\|\Phi_1\|^p_{L^p(\Omega,\omega_1)}\right],
\end{aligned}$$

then, using condition **(H1)**, we deduce, as $k \longrightarrow \infty$

$$Hu_{m_k}(x)) \longrightarrow Hu(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the Lebesgue's theorem, we obtain

$$\|Hu_{m_k} - Hu\|_{L^{p'}(\Omega,\omega_1)} \longrightarrow 0,$$

that is,

$$Hu_{m_k} \longrightarrow Hu \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

We conclude, from the convergence principle in Banach spaces, that

$$Hu_m \longrightarrow Hu \quad \text{in} \quad L^{p'}(\Omega,\omega_1). \tag{8}$$

Step 4:
We define the operator

$$\tilde{H} : W_0^{1,p}(\Omega,\omega_1) \longrightarrow L^{q'}(\Omega,\omega_2)$$
$$(\tilde{H}u)(x) = \mathcal{H}(x, u(x), \nabla u(x)).$$

We now show that the operator $\tilde{H}$ is bounded and continuous.

(i) Let $u \in W_0^{1,p}(\Omega,\omega_1)$. Using **(A2)** and Remaek 3.1, we obtain

$$\begin{aligned}
\|\tilde{H}u\|^{q'}_{L^{q'}(\Omega,\omega_2)} &= \int_\Omega |\mathcal{H}(x,u(x),\nabla u(x))|^{p'}\omega_2 dx\\
&\le \int_\Omega \left(K_3 + h_4|u|^{\frac{q}{q'}} + h_5|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\\
&\le C_q \int_\Omega \left[K_3^{q'} + h_4^{q'}|u|^q + h_5^{q'}|\nabla u|^q\right]\omega_2 dx\\
&\le C_q \Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\omega_2)}\\
&\quad + \|h_5\|^{q'}_{L^\infty(\Omega)}\|\nabla u\|^q_{L^q(\Omega,\omega_2)}\Big]\\
&\le C_q \Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)}\\
&\quad + \|h_5\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|\nabla u\|^q_{L^p(\Omega,\omega_1)}\Big]\\
&\le C_q \Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\Big(\|h_4\|^{q'}_{L^\infty(\Omega)}\\
&\quad + \|h_5\|^{q'}_{L^\infty(\Omega)}\Big)\|u\|^q_{W_0^{1,p}(\Omega,\omega_1)}\Big],
\end{aligned}$$

where the constant C_q depends only on q.

(ii) Let $u_m \longrightarrow u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m \longrightarrow \infty$. We need to show that $\tilde{H}u_m \longrightarrow \tilde{H}u$ in $L^{q'}(\Omega,\omega_2)$.
According to **(A2)** and Remark 3.1, we have

$$\begin{aligned}
&\|\tilde{H}u_{m_k} - \tilde{H}u\|^{q'}_{L^{q'}(\Omega,\omega_2)} = \int_\Omega |\tilde{H}u_{m_k}(x) - \tilde{H}u(x)|^{q'}\omega_2 dx\\
&\le \int_\Omega \Big(|\mathcal{H}(x,u_{m_k},\nabla u_{m_k})| + |\mathcal{H}(x,u,\nabla u)|\Big)^{q'}\omega_2 dx\\
&\le C_q \int_\Omega \Big(|\mathcal{H}(x,u_{m_k},\nabla u_{m_k})|^{q'} + |\mathcal{H}(x,u,\nabla u)|^{q'}\Big)\omega_2 dx\\
&\le C_q \Bigg[\int_\Omega \left(K_3 + h_4|u_{m_k}|^{\frac{q}{q'}} + h_5|\nabla u_{m_k}|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\\
&+ \int_\Omega \left(K_3 + h_4|u|^{\frac{q}{q'}} + h_5|\nabla u|^{\frac{q}{q'}}\right)^{q'}\omega_2 dx\Bigg]\\
&\le 2C_qC'_q \int_\Omega \left(K_3^{q'} + h_4^{q'}\Phi_1^q + h_5^{q'}\Phi_2^q\right)\omega_2 dx\\
&\le 2C_qC'_q \Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|h_4\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^q(\Omega,\omega_2)}\\
&+ \|h_5\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^q(\Omega,\omega_2)}\Big]\\
&\le 2C_qC'_q \Big[\|K_3\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\|h_4\|^{q'}_{L^\infty(\Omega)}\|\Phi_1\|^q_{L^p(\Omega,\omega_1)}\\
&+ C^q_{p,q}\|h_5\|^{q'}_{L^\infty(\Omega)}\|\Phi_2\|^q_{L^p(\Omega,\omega_1)}\Big].
\end{aligned}$$

Hence, from **(A1)**, we deduce, as $k \longrightarrow \infty$

$$\tilde{H}u_{m_k}(x) \longrightarrow \tilde{H}u(x), \quad \text{a.e. } x \in \Omega.$$

Therefore, by the the Lebesgue's theorem, we obtain

$$\|\tilde{H}u_{m_k} - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)} \longrightarrow 0,$$

that is,

$$\tilde{H}u_{m_k} \longrightarrow \tilde{H}u \quad \text{in} \quad L^{q'}(\Omega,\omega_2).$$

Thanks to convergence principle in Banach spaces, we conclude that

$$\tilde{H}u_m \longrightarrow \tilde{H}u \quad \text{in} \quad L^{q'}(\Omega,\omega_2). \tag{9}$$

Finally, let $\varphi \in W_0^{1,p}(\Omega,\omega_1)$ and using Hölder inequality, Theorem 2.2(with $\theta = 1$) and Remark 3.1, we obtain

$$\begin{aligned}
&|\mathbf{B}_1(u_m,\varphi)-\mathbf{B}_1(u,\varphi)| \\
&\quad = |\int_\Omega \langle \mathcal{A}(x,\nabla u_m)-\mathcal{A}(x,\nabla u),\nabla\varphi\rangle \omega_1 dx| \\
&\quad \le \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,\nabla u_m)-\mathcal{A}_j(x,\nabla u)||D_j\varphi|\omega_1 dx \\
&\quad = \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\omega_1 dx \\
&\quad \le \sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\|D_j\varphi\|_{L^p(\Omega,\omega_1)} \\
&\quad \le \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

$$\begin{aligned}
&|\mathbf{B}_2(u_m,\varphi)-\mathbf{B}_2(u,\varphi)| \\
&= |\int_\Omega \langle \mathcal{B}(x,u_m,\nabla u_m)-\mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle \omega_2 dx| \\
&\le \sum_{j=1}^n \int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m)-\mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\omega_2 dx \\
&= \sum_{j=1}^n \int_\Omega |G_j u_m - G_j u||D_j\varphi|\omega_2 dx \\
&\le \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right)\|\nabla\varphi\|_{L^q(\Omega,\omega_2)} \\
&\le C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right)\|\nabla\varphi\|_{L^p(\Omega,\omega_1)} \\
&\le C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

$$\begin{aligned}
&|\mathbf{B}_3(u_m,\varphi)-\mathbf{B}_3(u,\varphi)| \\
&\quad \le \int_\Omega |g(x,u_m)-g(x,u)||\varphi|\omega_1 dx \\
&\quad = \int_\Omega |Hu_m - Hu||\varphi|\omega_1 dx \\
&\quad \le \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)}\|\varphi\|_{L^p(\Omega,\omega_1)} \\
&\quad \le \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and

$$\begin{aligned}
&|\mathbf{B}_4(u_m,\varphi)-\mathbf{B}_4(u,\varphi)| \\
&\quad \le \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m)-\mathcal{H}(x,u,\nabla u)||\varphi|\omega_2 dx \\
&\quad = \int_\Omega |\tilde{H}u_m - \tilde{H}u||\varphi|\omega_2 dx \\
&\quad \le \|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}\|\varphi\|_{L^q(\Omega,\omega_2)} \\
&\quad \le C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Hence, for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$, we have

$$\begin{aligned}
&|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| \\
&\le |\mathbf{B}_1(u_m,\varphi)-\mathbf{B}_1(u,\varphi)| + |\mathbf{B}_2(u_m,\varphi)-\mathbf{B}_2(u,\varphi)| \\
&\quad + |\mathbf{B}_3(u_m,\varphi)-\mathbf{B}_3(u,\varphi)| + |\mathbf{B}_4(u_m,\varphi)-\mathbf{B}_4(u,\varphi)| \\
&\le \Big[\sum_{j=1}^n \Big(\|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\
&\quad + \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}\Big]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\|\mathbf{A}u_m - \mathbf{A}u\|_* \\
&\le \sum_{j=1}^n \Big(\|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\Big) \\
&\quad + \|Hu_m - Hu\|_{L^{p'}(\Omega,\omega_1)} + C_{p,q}\|\tilde{H}u_m - \tilde{H}u\|_{L^{q'}(\Omega,\omega_2)}.
\end{aligned}$$

Combining (6), (7), (8) and (9), we deduce that

$$\|\mathbf{A}u_m - \mathbf{A}u\|_* \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

that is, **A** is continuous.
Hence, the proof of the theorem 4.1 is completed.

V. Example

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$, and consider the weight functions $\omega_1(x,y) = (x^2+y^2)^{-1/2}$ and $\omega_2(x,y) = (x^2+y^2)^{-1/3}$ (we have that $\omega_1,\omega_2 \in A_4$, $p=4$ and $q=3$), and the functions $\mathcal{B}_j : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$, $\mathcal{A}_j : \Omega\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ $(j=1,2)$, $g : \Omega\times\mathbb{R}\times \longrightarrow \mathbb{R}$ and $\mathcal{H} : \Omega\times\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$A_j((x,y),\xi) = h_1(x,y)\xi_j^3,$$

where $h_1(x,y) = 2e^{(x^2+y^2)}$,

$$\mathcal{B}_j((x,y),\eta,\xi) = h_3(x,y)|\xi_j|\xi_j,$$

where $h_3(x,y) = 2 + sin(x^2+y^2)$,

$$g((x,y),\eta) = h_6(x,y)|\eta|^3 sgn(\eta),$$

where $h_6(x,y) = 2 - sin^2(x+y)$, and

$$\mathcal{H}((x,y),\eta,\xi) = h_5(x,y)\xi^2 sgn(\eta),$$

where $h_5(x,y) = 2 - cos^2(xy)$.
Let us consider the partial differential operator

$$\begin{aligned}
Lu(x) = -\mathrm{div}\Big[\omega_1(x)\mathcal{A}(x,\nabla u(x)) + \omega_2(x)\mathcal{B}(x,u(x),\nabla u(x))\Big] \\
+ \omega_1(x)g(x,u(x)) + \omega_2(x)\mathcal{H}(x,u(x),\nabla u(x)),
\end{aligned} \tag{10}$$

Therefore, by Theorem 4.1, the problem

$$\begin{cases} Lu(x,y) = \frac{cos(xy)}{(x^2+y^2)} - \frac{\partial}{\partial x}\left(\frac{sin(xy)}{(x^2+y^2)}\right) - \frac{\partial}{\partial y}\left(\frac{sin(xy)}{(x^2+y^2)}\right) & \text{in } \Omega, \\ u(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$

admits one and only solution $u \in W_0^{1,4}(\Omega,\omega_1)$.

REFERENCES

[1] A. Abbassi, E. Azroul, A. Barbara, Degenerate $p(x)$-elliptic equation with second membre in L^1, Advances in Science, Technology and Engineering Systems Journal. Vol. 2, No. 5, 45-54, 2017.
[2] A. Abbassi, C. Allalou and A. Kassidi. Existence of Entropy Solutions for Anisotropic Elliptic Nonlinear Problem in Weighted Sobolev Space. In : The International Congress of the Moroccan Society of Applied Mathematics. Springer, Cham, p. 102-122, 2019.
[3] L. Aharouch , E.Azroul and A.Benkirane, Quasilinear degenerated equations with L^1 datum and without coercivity in perturbation terms. Electronic Journal of Qualitative Theory of Differential Equations, vol no 19, p. 1-18, 2006.
[4] Y. Akdim and Chakir Allalou. "Existence and uniqueness of renormalized solution of nonlinear degenerated elliptic problems." Analysis in Theory and Applications 30, 318-343, 2014.
[5] L. Boccardo and T. Gallouëtt, Strongly nonlinear elliptic equations having natural growth terms and L^1, Nonlinear Analysis Theory methods and applications, 19(6), 573-579, 1992.
[6] L. Boccardo, T. Gallouët, and L. Orsina, Existence and nonexistence of solutions for some nonlinear elliptic equations,Journal Analyse Mathematique, vol. 73, pp. 203-223, 1997.
[7] H. Brezis and W. Strauss, Semilinear second-order elliptic equations in L^1, J. Math. Soc. Japan, 25(4), 565-590, 1973.
[8] A.C. Cavalheiro. Existence of solutions for some degenerate quasilinear elliptic equations. Le Matematiche, vol. 63, no 2, p. 101-112, 2008.
[9] A.C. Cavalheiro. Existence Results For A Class Of Nonlinear Degenerate Elliptic Equations. Moroccan Journal of Pure and Applied Analysis, vol. 5, no 2, p. 164-178, 2019.
[10] Chiarenza, Filippo. "Regularity for solutions of quasilinear elliptic equations under minimal assumptions." Potential Theory and Degenerate Partial Differential Operators. Springer, Dordrecht, 325-334, 1995.
[11] Drbek, Pavel, Alois Kufner, and Francesco Nicolosi. Quasilinear elliptic equations with degenerations and singularities. Vol. 5. Walter de Gruyter, 2011.
[12] Fabes, Eugene B., Carlos E. Kenig, and Raul P. Serapioni. "The local regularity of solutions of degenerate elliptic equations." Communications in Statistics-Theory and Methods 7.1, 77-116, 1982.
[13] S. Fucik, O. John, and A. Kufner "Function Spaces, Noordhoff International Publishing, Leyden. Academia, Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977."
[14] García-Cuerva, José, and J.L. Rubio De Francia. Weighted norm inequalities and related topics. Elsevier, North-Holland Mathematics Studies 116, Amsterdam, 1985.
[15] J. Heinonen, T. Kilpelainen and O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford Math. Monographs, Clarendon Press, Oxford,1993.
[16] Kufner, Alois. Weighted sobolev spaces. Vol. 31. John Wiley and Sons Incorporated, 1985.
[17] Kufner, Alois, and Bohumìr Opic. "How to define reasonably weighted Sobolev spaces." Commentationes Mathematicae Universitatis Carolinae 25.3, 537-554, 1984.
[18] Muckenhoupt, Benjamin. "Weighted norm inequalities for the Hardy maximal function." Transactions of the American Mathematical Society, 207-226, 1972.
[19] M. Rủzicka, Electrorheological Fluids: Modelling and Mathematical Theory, Springer, Berlin, 2000.
[20] Turesson, Bengt O. Nonlinear potential theory and weighted Sobolev spaces. Vol. 1736. Springer Science and Business Media, 2000.
[21] Xu, Xiangsheng. "A local partial regularity theorem for weak solutions of degenerate elliptic equations and its application to the thermistor problem." Differential and Integral Equations 12.1, 83-100, 1999.
[22] E. Zeidler, Nonlinear Functional Analysis and its Applications, vol.II/B, Springer-Verlag, New York,1990.

Performance and analyses using two ETL extraction software solutions

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Abstract—

In the prospect of doing a set of decision support onboards in a public university, we will present a comparison of two ETL extraction based in a production databases of students' information. For the deployment, we use Pentaho and Sql Server Tools and we demonstrate the application on the case of Sultan Moulay Slimane University in Beni Mellal, Morocco

Keywords—*:* Pentaho; Sql Server; Data Warehouse; Business Intelligence

I. INTRODUCTION

Data warehouse (DWs) is delineated as "subject-oriented, more integrated, timely-variant, and non-volatile collection of data to support the management decision process" [1]. Data warehouse emphasizes the collection of data from multiple sources for useful analysis.

At the center of DWs is the extraction-transformation-loading (ETL) process. ETL is a process utilized to extract data from multiple sources, transform that data to the desired state through cleansing, and load it into a target database. The deliverable is used to generate reports and for analysis. ETL consumes up to 70% of all the resources [2-5].

In the most professional field, the main approach before selecting an ETL tool is to perform proofs of concept. However, it is almost impossible to perform proofs of concept of all ETL tools available on the market. Then a pre-selection is made in the way that two ETL suites are kept for testing. This pre-selection is generally based on criteria summarized as follows: the category of the tool, the cost, the type of ETL project, and the proof of concepts.

In this white paper, we will only look at the use of two ETL tools (Microsoft SQL Server Integration Services SSIS and Pentaho Kettle) [6] based on the generalized criteria for selecting the better tool.

II. RELATED WORK

In the recent years, a number of different approaches have been suggested for the design, optimization, and automation of ETL operations. In this section, we present a brief overview of these several approaches [7]. Some of the leading data integration vendors are IBM, Informatica, Oracle, Microsoft, Talend, Pentaho, Information Builders, etc.

There are many available research papers that offer a comparative view of the leading ETL tools in the market, such as [8-9]. They analyze in details the functionalities and features offered by these tools, and it can be deduced that all of them provide support for all the features that define data integration tools.

Different variants of some approaches for integration of ETL tools with data warehouses have been proposes. Shaker H. Ali ElSappagh tries to navigate through the efforts that have been made to use acronyms for ETL, DW, DM, OLAP, Ion-line analytical processing.A data warehouse gives a set of numerical values based on a set of input values in the form of dimensions [10]. Li, Jain, overcame the limitations of the traditional architecture of Extract, Transform, Load tools, and developed a three-layer architecture based on metadata. This made the ETL process more flexible, versatile and efficient, and finally they designed and implemented a new ETL tool for the drilling data warehouse [11]. A systematic review method was proposed to identify, extract, and analyze the main proposals for modeling the conceptual ETL process for data warehouse. The main proposals were identified and compared based on the characteristics, activities, and notation of ETL processes, and the study was concluded by reflecting on the studied approaches and providing an update skeleton for future studies.

III. Feature Comparison beween PDI and SSIS

In this section, we are going to do a comparative study of the features for the two extraction tools, especially the Pentaho Data Integration and the Microsoft SQL Server Integration Services

A. Access to data

Table 1. Access to data

features	PDI	SSIS
Read the full table	✓	✓
Complete view of reading	✓	✓
Calling stored procedure	✓	✓
Uploading clause where/order by	✓	✓
Query	✓	✓
Query Builder	✓	✓
Reading / writing all simple and complex data types	✓	✓
Read the full table	✓	✓
CSV	✓	✓
Fixed / Limited	✓	✓
XML	✓	✓
Excel	✓	✓
Validity flat files	x	✓
Validity of XML files	✓	✓

For the access to relational data, flat files and applications of connectors, PDI and SSIS are good solutions for thesefeatures.The two tools allow the analysis of data from various sources to determine the transformations necessary to perform aggregations, data deletions, automatic corrections of errors, etc.But for the validation of the flat files, the SSIS tool is more robust in comparaison to PDI.

B. Triggering pocess

Table 2. Triggering process

features	PDI	SSIS
CORBA	x	✓
XML RPC	x	✓
JMC	x	x
MOMS	x	✓
Index	✓	✓
POP	✓	✓

We note for the triggering process by message, the PDI tool is not suitable for this procedure, whereas for the trigger by type of polling the two tools are robust.
Oracle is the only database that supports JMS natively in the form of Oracle Advanced Queueing. If the message receiver is not tookeen on thisJMS implementation, it is usually possible to find some sort of messaging bridge that will transform and forward messages from one JMS implementation to another.

C. Data processing

Table 3. Data processing

Features	PDI	SSIS
Transformation functions of dates and numbers	✓	✓
Statistical functions qualities	x	✓
Allows transcoding with a reference table	x	✓
Heterogeneous joints	x	✓
Supported modes of joint	external	✓
Management of nested queries	x	✓
Treatment options for a programming language	✓	✓
Added new transformations and business processes	✓	✓
Mapping graphics	✓	✓
Drag and Drop	✓	✓
Graphical representation of flow	✓	✓
Viewing under development data	x	✓
Impact analyses tools	✓	✓
Debugging Tools	✓	✓
Generation of technical and functional documentation	x	✓
Viewing documentation through the web	x	✓
Management of integration errors	For some steps	✓

The two tools provide a mechanism of query directly in SQL which allows to make all modes of joint and nested queries.It ispossible with SQL Server to join data from an active directory to data in a SQL Server and create a view of the joined data. For the treatment of the data, the two tools are not compatible for the transformations and calculations by default, they are recommended for the manual transformations except for the generation of technical and functional documents.

D. Advanced development and deployment/production start

Table 4.Advanced development and deployment/production start

Features	PDI	SSIS
Application Programming Interface	✓	✓
Integration of external functions	✓	✓
Crash recovery mechanism	x	x
Setting buffers / indexes / caches	✓	✓
Team Development Management	✓	✓
Versioning	x	✓
Compilation treatments	x	Yes for C#
Type into production	Windows or Unix command line	Windows command line
History visualization into production	x	x

It was found that the two tools are not compatible for the recovery mechanism on incident and for the history visualization into production,but generally they are used for the other properties of the advanced development and deployment of production setting.

E. Administration and security management

Table 5.Administration and security management

Features	PDI	SSIS
Administration Console	✓	✓
Automated log management	✓	✓
Specific log generation	x	✓
Interfacing with monitoring tools	x	✓
Integrated treatment planning tool	x	✓
Use of rights of a directory	x	x
Security type	DBMS security which contains the repository	✓
Security scenario creation	✓	✓
Security access to metadata	✓	✓
Safety manual task launch	✓	✓
Security Administration Console	✓	✓

We note that the PDI is not compatible for the generation of specific log, the interfacage with Tools of Supervision, the planning of integrated treatment and for the security of the database management system that does not contain the repository.

IV. COMPARATIVE TREATMENT TIMES

A. Test realization methodology

Test n°

Descriptive

1. Extracting data from an Excel file
2. Loading data into another Excelfile
3. The input file contains 5 typed fields:
    - COD_IND [NUMBER] (Student Code)
    - COD_NNE_IND [NUMBER] (National ID of the student)
    - DATE_NAI_IND [DATE] (Date of birth of the student)
    - LIB_NOM_PAT_IND [String] (Family name of student)
    - LIB_PR_IND [String] (Student's first name)

B. Modeling in Pentaho Data Integration (PDI) [8]. [9].

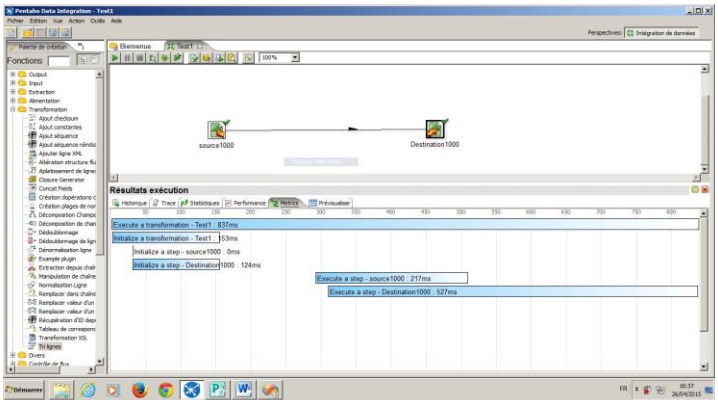

Fig. 1: Extraction of 1000 rows with PDI

C. Modeling in SQL Server Integration Services (SSIS)[10].

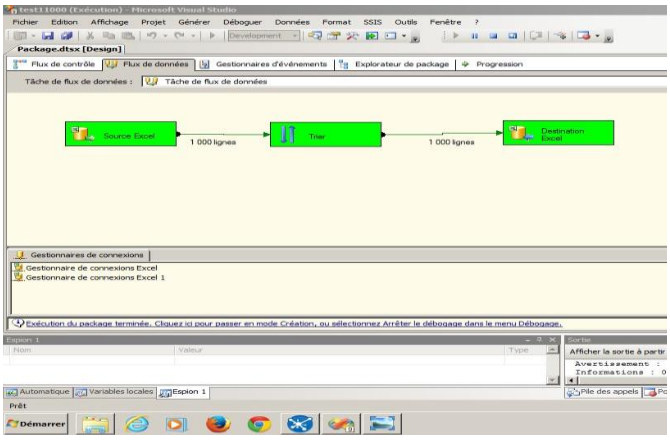

Fig. 2: Extraction of 1000 rows with SSIS

We performed the same work for 5000 and 10000 rows.

Table 6.Processing time for both tools

Number of rows	PDI	SSIS
1000	837	655
5000	1384	1014
10000	3009	1513

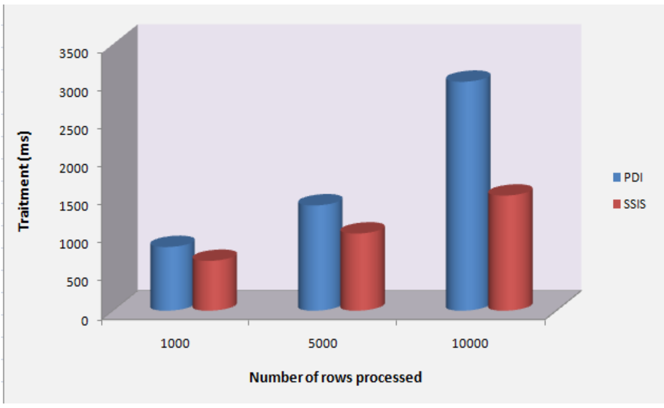

Fig3: Comparison of the results obtained for the two tools

The performance of the treatment of time is an important criterion in the choice of an ETL, but from these results we cannot prejudge the actual performance in a production environment, since time of execution variesfollowing the typology of treatments.

At the end of our comparative study, we can conclude that SSIS and PDI are two tools of ETL with their own specificities.These are real alternatives to the ETL owners as Informatica Power Centeror Oracle Warehouse Builder.These two tools offer all the features necessary for an ETL.

V. 5. Conclusion

Both SSIS and PDI are robust solutions to perform ETL in a data warehouse. SSIS emphasizes configuration over coding; however, because of the limited amount of available transformation objects, coding will be required to process complex data. SSIS's strength comes from its control flow, data flow and event driven architecture. It allows great flexibility to the developer to design the structure and flow the ETL process.On the other side, PDI includes many more options to access outside data such as a Google Analytics and several options to access Web services. It can be used on either Windows or Linux operating systems.

The choice between the SSIS ETL and PDI thus depends essentially on the typology of the project it leads.

References

[1] Inmon W, Strauss D, Neushloss G. DW 2.0 the Architecture for the next generation of Data Warehousing. Morgan Kaufman. 2007.

[2] Simitisis A, Vassiliadis P, Skiadopoulos S, Sellis T. Data Warehouse Refreshment. Data Warehouses and OLAP: Concepts, Architectures and Solutions. IRM Press. 2007: 111-134.

[3] Kimball R, Caserta J. The Data Warehouse ETL Toolkit: Practical Techniques for Extracting, Cleaning, Conforming, and Delivering Data. Wiley Publishing, Inc. 2004.

[4] Kabiri A, Wadjinny F, Chiadmi D. Towards a Framework for Conceptual Modelling of ETL Processes. Proceedings of The first international conference on Innovative Computing Technology (INCT 2011), Communications in Computer and Information Science. Heidelberg. 2011; 241: 146-160.

[5] Vassiliadis P, Simitsis A. Extraction-Transformation-loading, (ETL) Processes in Data Warehouse Environments. In Encyclopedia of Database Technologies and Applications, Idea Group. 2005.

[6] Golfarelli M, Rizzi S. Data Warehouse Design, Modern Principles and Methodologies. McGraw Hill Companies, srl Publishing Group Italia. 2009.

[7] Jovanovic P, Theodorou V, Abelló A, Nakuçi E. Data generator for evaluating ETL process Quality (Science direct). In Press. 2016.

[8] Thoo E, Friedman T, Beyer Mark A. Magic Quadrant for Data Integration Tools. Gartner RAS Core Research Note G. 2013.

[9] Pall AS, Khaira JS. A comparative review of extraction, transformation and loading tools. Database Systems Journal BOARD. 2013; 4(2): 42-51.

[10] Simitsis A, Vassiliadis P, Sellis T. State space optimization of ETL workflow. IEEE Transactions on Knowledge and Data Engineering. 2005; 17(10): 1404-1419.

[11] Jian L, Bihua X. ETL tool research and implementation based on drilling data warehouse. Seventh International Conference on Fuzzy Systems and Knowledge Discovery. 2010; 6: 2567-2569.

A novel C/Ag/SiO2 Sonogel-Carbon electrode

Preparation, characterization, application and statistical validation as amperometric sensors for cathecol determination

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***Abstract*— The monitoring of the cathecol level is clinically important. In this work a novel C/Ag/SiO2 Sonogel-Carbon electrode was used for the sensitive voltametric determination of cathecolamine. A complete characterization of the electrodes has been performed using scanning electron microscopy, Raman spectroscopy, cyclic voltammetry, and impedance spectroscopy. the novel electrode has shown an increase in the effective area of up to 70%, oxidation peaks and an excellent electrocatalytic activity. The electrochemical response characteristics were investigated by cyclic and differential pulse voltammetry, the limit of detection is estimated to be in the sub micromolar regime.**

statistical analysis of measurements performed in water samples has led to good apparent recovery.

Keywords*— *The statistical analysis, C/Ag/SiO2 Sonogel-Carbon electrode. Amperometric sensor. Cathecol.

I. Introduction

The application of sensors for clinical measurement are well recognised in the last ten years. In this work a C/Ag/SiO2 Sonogel-Carbon electrode is used for the sensitive pulse voltammetry determination of cathecol. The proper choice of the sensing material, in view of the specific application, is fundamental since it can impart to the device definite physicochemical properties and analytical peculiarities. The main advantages sought by adopting a specific electrode material are the lowering of the potentials at which charge transfer processes occur, the enhancement of the rele vant current and the prevention of the passivation of the surface. The results of this paper have shown an analytical performance and an efficient catalytic activity of the electrode for the electro-oxidation of cathecol. The advantage of functional materials as an immobilization matrix for sensors is due to high surface to volume ratio, the presence of reactive groups on the surface, and fast electron transfer kinetics [1]. In recent years, nanostructured materials gained a very important role in the development of amperometric sensors [2]. The high superficial area/volume ratio and the polyhedral shape induce a quite high number of defects to be present at the electrode surface, imparting to the material high reactivity toward species in solution, suitable for the realization of electrocatalytic processes [3]. The result of our proposed modified electrode as compared to other electrochemical methods reported in the literature [4,8] exhibit that our electrochemical sensor Seems to be very promising and they can be considered for quantification of cathecol.

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Modified materials	Detection limit (µM)	Ref
N-doped carbon nanotubes	0.02	[4] Q. Gou et al., 2016
Reduced graphene oxide	0.18	[5] H. Zhang et al., 2015
Glassy carbon electrode modified with graphene	0.01	[6] H. Du et al., 2011
Activated carbon	0.05	[7] H.Hammani et al., 2019
KOH-activated graphene sheets carbon film	0.1	[8] L. Huang et al., 2016
$C/Ag/SiO_2$ Sonogel-Carbon	0.01	This work

II. Experimental

A. *Reagents and materials*

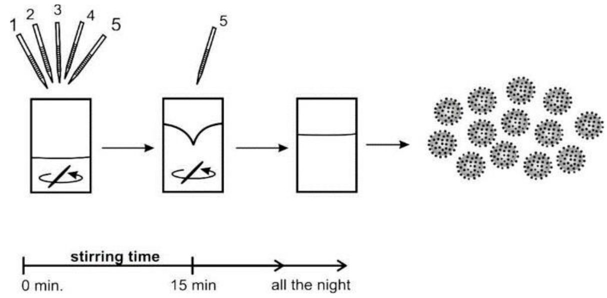
Cathecol reagent grade ⩾98% (HPLC) is purchased from Sigma Aldrich (USA). KH2PO4, K2HPO4 for phosphate buffer and graphite powder (<20 microns) were purchased from Fluka. Paraffin oil was purchased from a pharmacy. All other chemicals were of reagent grade and used directly without further purification. Plastic capillary tubes, i.d. 2mm, were used as the bodies for the composite electrodes. Solutions were prepared using deionized double-distilled water with a measured resistance higher than 15 µS cm−1.

B. *Instrumentation*

The cyclic voltammetry (CV), differential pulse voltammetry (DPV) and electrochemical impedance (EI) were applied to study the behaviour of C/Ag/SiO2 Sonogel-Carbon electrode. They were all performed with a Voltalab®40, type PGZ301 from Radiometer (France). A conventional three-electrode cell (20 mL) was used at room temperature (25± 1°C), the counter electrode was a platinum wire and an SCE, 3M KCl electrode was used as the reference, the C/Ag/SiO2 electrodes were used as working electrode. The scanning electron microscope (SEM) image was obtained using a HITACHI X-650 SEM instrument. The statistical validation was carried out by the MATLAB statistical software.

C. *Preparation of the C/Ag/SiO2 Sonogel-Carbon electrode*

To prepare the C/Ag/SiO2, the following procedure was used: 0.1g of silver nanoparticles silica Ag/SiO2 (Scheme 1) is dispersed in 0.5 M acetic acid solution, then 1g of Carbon graphite powder was dispersed in the solution until obtaining a unique phase, and then the mixture was heated at 120°C to evaporate the acetic acid and water. In the next step, the carbon powder modified with Ag/SiO2, is dried and was mixed thoroughly in a mortar with 40% of paraffin oil until obtaining a homogenous paste. Thus, the plastic capillaries were filled, leaving a little extra mixture sticking out of the tube to facilitate the subsequent polishing. For establishing electrical contact, a copper wire was inserted into the capillary. Before usage the electrodes were polished with emery paper No1500, and were electrochemically cleaned by cyclic voltammetry until obtaining a stable cyclic voltammograms between -0,80 and 1,50 V in 0,005 mol.L-1 KCl.



Scheme 1. Procedure for depositing silver nanoparticles on silica spheres:

1. TEOS, 2. ethanol, 3. ammonia, 4. water, and 5. Ag NPs.

III. Results and Discussion

A. *Surface and Electrochemical characterization of modified electrodes*

Scanning microscopy (SEM) was used to explore the difference in structure between films of carbon paste alone, and those of carbon paste in the presence of SiO2 (C / SiO2), and mixed compound of SiO2 and Ag (C / SiO2 / Ag).

Analysis of the surface of the carbon-only paste electrode shows a granular structure of carbon. However, the incorporation of SiO2 into the carbon paste shows a more structured surface (cauliflower-shaped) with the appearance of bright white particles. An even better organized morphology, when silver is added (C / Ag / SiO2 electrode), is noted, corresponding to the C paste modified by silica-silver nanoparticles. In this case, the film generated shows a better organization and an oriented structure as well as an increase in the specific surface (Figure 1). The presence of SiO2 and Ag in the carbon paste therefore seems to have a favorable effect on the structure of the materials prepared.

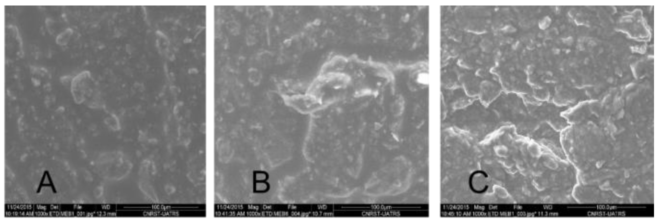


Figure 1. (A) SEM images obtained for C electrode. (B) SEM images obtained for C/SiO_2 modified electrode. (C) SEM images obtained for $C/Ag/SiO_2$ modified electrode

To characterize the interface features of the modified electrode surface we have used the EIS method. The Figure 2 shows that the charge transfer resistance of C/Ag/Si O_2 electrode is much smaller than that of C/Si O_2 electrode and the C electrode, suggesting that it is easier to transfer electrons at $C/Ag/SiO_2$, and this indicates that the incorporation of silver nanoparticles on silica spheres promote the electron transfer synergistically and accelerates the diffusion of ferricyanide towards the modified electrode surface.

The active surface area of the modified electrode was estimated according to the slope of the i_p versus $v^{1/2}$ plot, based on the Randles–Sevcik equation [9,10]:

$$Lp = 2.69 \times 10^5 n^{3/2} A_{eff} D^{1/2} C \upsilon^{1/2}$$

Where A_{eff} is the effective surface area, n is the number of electrons transferred, D (= 7.6 × 10^{-6} cm^2 s^{-1}) is the diffusion coefficient of potassium ferricyanide (Gooding et al., 1998), and C are the concentration of potassium ferricyanide. The effective electrode area for C/Ag/SiO_2 modified electrode is approximately 0.058 cm^2 whereas 0.037cm^2 for C/SiO_2 and 0.032cm^2 for C electrode.

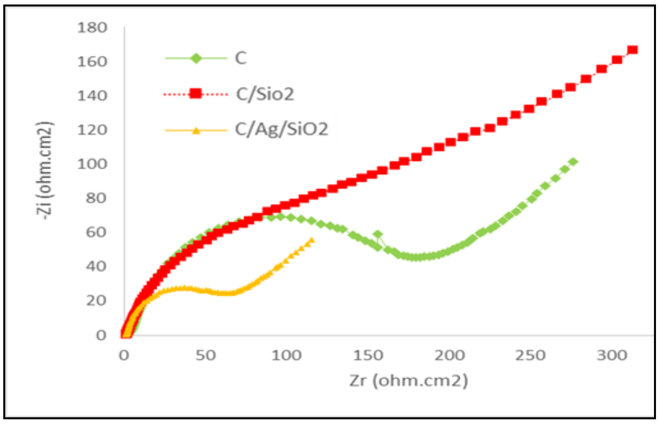
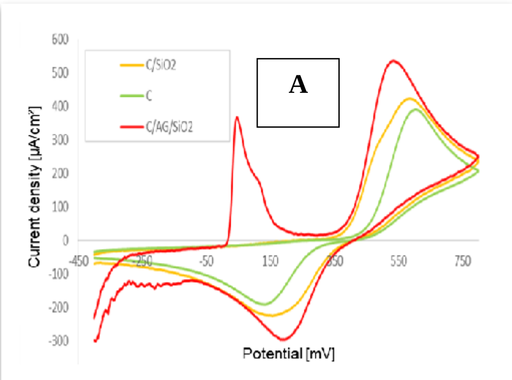
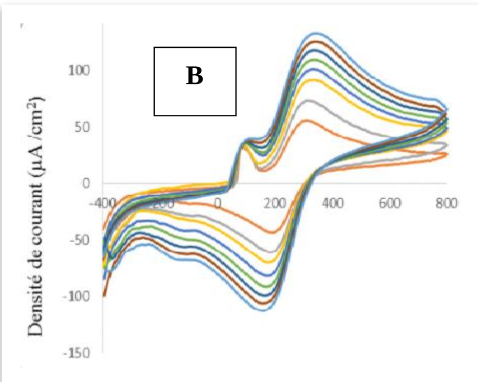


Figure 2. Nyquist plots of EIS in 10^{-2} M potassium ferricyanide prepared in 0.05 M KCl for C/Ag/SiO_2 electrode and C/SiO_2 electrode and Carbon electrode, Amplitude: 5 mV; frequency range: 100 kHz–10 mHz; potential: 0V.

B. Electrochemical Behaviour of cathecol at the Modified Electrode Use an unique style for units.





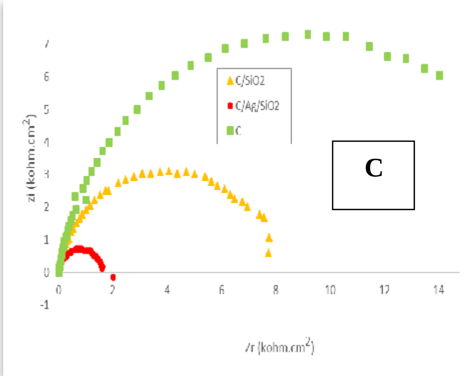


*Figure 3. **(A)** cyclic voltammetry of 10^{-5}M of cathecol at C/Ag/SiO_2 electrode and C/SiO_2 electrode and Carbon electrode in PBS (0.05M), pH = 2, T = 25 ° C. **(B)** Cyclic voltammograms obtained at different scan rates from the C/Ag/SiO_2 modified electrode in a PBS at pH 2 containing 2 µM of cathecol. Scan rates: 40, 60, 80,100, 120, 140,160 and 180 mV/s, **(C)** Nyquist plots of 10^{-5}M of cathecol at C/Ag/SiO_2 electrode and C/SiO_2 electrode and Carbon electrode in PBS (0.05M), pH = 2, T = 25 ° C.*

The Figure 3.**A**, shows the electrochemical behaviour of the cathecol at C/Ag/SiO_2 electrode and C/SiO_2 electrode and Carbon electrode in PBS pH 2 using CV; First, the cathecol (pK_a =9,5) presents an electroactive character that appears with an oxidation peak in the studied potential ranges **(Tables.1),** also we noticed the appearance of Ep_a peak corresponding to the oxidation of Ag incorporated in the paste of the modified electrode at 100 (mV)/ECS. The relationship between the oxidation peak current (i_{pa}) and the square root of the scan rate ($v^{1/2}$) (Figure 3.**B)** is linear with linear correlation coefficients R = 0,9974, indicates that the electrochemical process is controlled by diffusion.
The (Figure 3.**C)**, shows the Nyquist plots behaviour of the cathecol at C/Ag/SiO_2 electrode and C/SiO_2 electrode and Carbon electrode in PBS pH 2, the **(Tables.2)** presents the charge transfer resistance and the capacitance of the electrical layer at the electrode/solution interface, and the apparent rate of electron transfer at different modified electrodes.

Electrode	Ip_a (µA/cm^2)	Ip_c (µA/cm^2)	Ip_a/Ip_c	Ep_a (mV)/ECS	Ep_c (mV)/ECS	ΔEp (mV)/ECS
Carbon electrode	263.513	-136.987	1.92	592	138	454
C/SiO_2 electrode	254.479	-147..27	1.72	570	178	392
C/Ag/SiO_2 electrode	373.19	-203.091	1.83	506	200	306

Tables.1 Electrochemical characterization of cathecol on three types of electrodes

Electrode	**R_{tc} (kohm .cm^2)**	**C_{dc} (µf/cm^2)**	**K_{app} (cm /s)**
Carbon electrode	17.04	10.45	$1.56.10^{-5}$
C/SiO_2 electrode	7.239	19.69	$3.68.10^{-5}$
$C/Ag/SiO_2$ electrode	1.507	23.65	$1.77.10^{-4}$

Tables.2 the electrical parameters of the three types of electrodes

From the table (**Tables.2)** we notice that the charge transfer resistance (Rtc) decreases for the Ag / SiO2 carbon paste electrode, a remarkable increase in the capacitance of the electric layer (Cdc) and an apparent speed increase of the electron transfer These results show the efficiency of Ag / SiO2 carbon modified electrode.

Given the results obtained in **Figure 3** and **Tables 1 and 2**, the presence of SiO2 and Ag in the carbon paste therefore seems to have a favorable effect on the structure of the materials prepared. The modified electrode should promote the sensitivity and the selectivity of determination cathecol. As a result, C/Ag/SiO2 can accelerate the electron transfer and decrease the overpotentials of cathecol oxidation at different levels of difusion modes, which is the key factor to adjust the problem of adsorptionat the electrod surface and realize determination directe of cathecol.

3.3. Analytical Calibration Curves of determinations of cathecol.

The DPV was used to obtain the calibration curve of cathecol at the modified electrode in PBS pH2. the result in Figure 4 shows the linear relationship between the oxidation peak current and cathecol concentrations. The peaks intensities are increased linearly in the range of 1–120 µM, the equation is Ipa (µA) = 0.3 C + 0.5 with a correlation coefficient of R^2 = 0,9992 and the detection limit (S/N = 3) estimated to be 0.01 µM in terms of signal to noise ratio of 3:1.

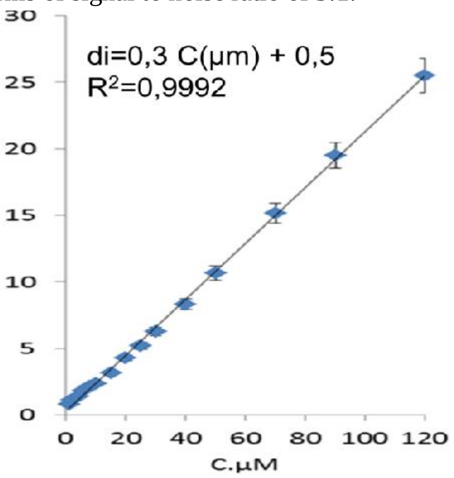


Figure 4. Calibration plots of cathecol (from 1 to 120 µmol L^{-1})

3.4 Determination of cathecol at $C/Ag/SiO_2$ in Urine.

The objective of this study is the simultaneous detection of the cathecol in the presence of AA and AU in the urine. In this context, and in order to evaluate the applicability of the proposed method for the determination of the cathecol and AA and AU in urine, the measurements were conducted in urine samples diluted 500% (with 0,05M PBS at pH 2) was then added to this mixture a deferred reports of AA, AU and cathecol (table 4).

the AA and UA is the principal organic constituents of urine, the phenomenon of interference on the electrochemical response of cathecol in the presence of the urinary AA and UA is one of the major problems that hinders electrochemical detection of its substances in biological media, since the unmodified carbon electrode could not separate cathecol and AA and UA oxidation peaks. The development of a simple and inexpensive device for the simultaneous determination and separation of the electrochemical responses of these substances remains the challenge of this work.

The Tables.4 shows that the peak currents for cathecol increase linearly with increases their respective concentrations, without considerable effects on the other peak currents of AA and UA while varying the concentration of cathecol from 10 to 100 µmol L^{-1}.

In addition, a various concentrations of AA from 20 to 100 µmol L^{-1} in the presence of cathecol and UA exhibit excellent responses to AA, AU, and cathecol without any obvious intermolecular effects among them, the peak current of AA increased linearly with increased concentration. is also indicated that the peak current of UA increased linearly with increases concentration of UA, without considerable effects on the other peak currents while varying the concentration of UA from 30 to 40 µmol L^{-1}.

These results confirm that the oxidation processes of cathecol, AA, UA at C/Ag/SiO2 electrode are independent from each other, this separation allows a simultaneous determination of AA, UA and cathecol in a mixture. The C/Ag/SiO2 possessed a higher active surface area and can separate cathecol and AA and UA oxidation peaks.

Table 3:
Simultaneous determination of cathecol, AA and UA in mixtures synthesis samples (±SD; the standard deviation for n=3).

Sample	**Added (µmol/L)**			**Found (µmol/L)**			**Recovery (%)**		
	CATHECOL	**AU**	**AA**	**CATHECOL**	**AU**	**AA**	**CATHECOL**	**AU**	**AA**
1	10	30	20	9.86±0.2	31±1.5	19.6±0.5	98.0%	103.3%	98.0%
2	30	40	40	28.4±1.3	41±2.7	38.3±1.5	94.7%	102.5%	95.8%
3	70	35	80	68.47±1.8	33.2±0.8	78.2±2.1	97.8%	100.6%	97.8%
4	100	40	100	98.5±1.7	40.3±1.3	98.7±1.2	98.5%	100.8%	98.7%

The feasibility of the C/Ag/SiO_2 sensor is demonstrated for analytical application, the recovery test was performed by the standard addition method (Table 3), with 4 different additions of cathecol, AA and UA to the urine diluted samples, the obtained recoveries ranged from 94.7 to 98.5 for cathecol; 95.8 to 98,7 for AA and 100,6 to 103,3 for UA. This high recovery and the perfect selectivity exerted by our C/Ag/SiO2 electrochemical sensor looks very promising for the simultaneous detection of cathecol, AA and AU. So, an effective sensor has been obtained for cathecol determination in urine sample in this work.

C. Conclusions

The use of electrochemical techniques for the sensitive and selective determination of cathecol in urine by differential pulse voltammetry using C/Ag/SiO_2 modified electrode was shows that the C/Ag/SiO_2 modified electrode present a perfect selectivity on the detection of the cathecol in presence of AA and UA in 0,05M PBS at pH 2, with a detection limit 0,01 µmol L^{-1} is obtained. This selectivity is maintained when the study is conducted in biological fluids such as in urine diluted 500% with 0,05M PBS at pH 2. Indeed, the results obtained were validated by the statistical validation methods and our electrochemical sensor looks very promising and they can be considered for early quantification of cathecol in clinical preparations.

[1] L. Pigani, C. Rioli, D. López-Iglesias, C. Zanardi, B. Zanfrognini, L.M. Cubillana-Aguilera, J.M. Palacios-Santander., 2020 Preparation and characterization of reusable Sonogel-Carbon electrodes containing carbon black: Application as amperometric sensors for determination of cathecol. Journal of Electroanalytical Chemistry. 877(2020) 114653. http://dx.doi.org/10.1016/j.jelechem.2020.114653

[2] Gooding, J.J., Praig, V.G., Hall, E.A.H., 1998. Platinum-Catalyzed Enzyme Electrodes Immobilized on Gold Using Self-Assembled Layers. Anal. Chem. 70, 2396–2402. doi:10.1021/ac971035t

[3] Maringa, A., Mugadza, T., Antunes, E., Nyokong, T., 2013. Characterization and electrocatalytic behaviour of glassy carbon electrode modified with

nickel nanoparticles towards amitrole detection. J. Electroanal. Chem. 700, 86–92. doi:10.1016/j.jelechem.2013.04.022

[4] Q. Gou, M. Zhang, G. Zhou, L. Zhu, Y. Feng, H. Wang, B. Zhong, H. Hou, Highly sensitive simultaneous electrochemical detection of hydroquinone and cathecol with threedimensional n-doping carbon nanotube film electrode, J. Electroanal. Chem. 760 (2016) 15, https://doi.org/10.1016/j.jelechem.2015.11.034.

[5] H. Zhang, X. Bo, L. Guo, Electrochemical preparation of porous graphene and its electrochemical application in the simultaneous determination of hydroquinone, cathecol, and resorcinol, Sensors Actuators B Chem. 220 (2015) 919, https://doi.org/10.1016/j.snb. 2015.06.035.

[6] H. Du, J. Ye, J. Zhang, X. Huang, C. Yu, A voltammetricsensor based on graphenemodified electrode for simultaneous determination of cathecol and hydroquinone, J. Electroanal. Chem. 650 (2011) 209, https://doi.org/10.1016/j.jelechem.2010.10. 002.

[7] H. Hammani, F. Laghrib, S. Lahrich, A. Farahi, M. Bakasse, A. Aboulkas, M.A. El Mhammedi, Effect of activated carbon in distinguishing the electrochemical activity of hydroquinone and cathecol at carbon paste electrode, Ionics 25 (2019) 2285, https://doi.org/10.1007/s11581-018-2648-6

[8] L. Huang, Y. Cao, D. Diao, Electrochemical activation of graphene sheets embedded carbon films for high sensitivity simultaneous determination of hydroquinone, cathecol and resorcinol, Sensors Actuators B Chem. 305 (2020) 127495, https://doi.org/10. 1016/j.snb.2019.127495

[9] H.-Y. Tseng, V. Adamik, J. Parsons, S.-S. Lan, S. Malfesi, J. Lum, L. Shannon, B. Gray, Development of an electrochemical biosensor array for quantitative polymerase chain reaction utilizing three-metal printed circuit board technology, Sensors Actuators B Chem. 204 (2014) 459–466. doi:http://dx.doi.org/10.1016/j.snb.2014.07.123.

[10] D. Salinas-Torres, F. Huerta, F. Montilla, E. Morallón, Study on electroactive and electrocatalyticsurfaces of single walled carbon nanotube-modifiedelectrodes, Electrochim. Acta. 56 (2011) 2464–2470. doi:http://dx.doi.org/10.1016/j.electacta.2010.1 1.02

Renewable energy
A new avenue for optimizing costs for companies

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***Abstract*—IJOA Journal**

Nowadays, Companies are carrying heavier and heavier loads than before, considering the market requirement, competitors who have become more numerous in almost all sectors of activity, and above all more creative and stronger on several fronts: marketing, marketing, productivity ... To win in the market, strong companies must have a strong management system that helps them optimize their costs and differentiate themselves from others to ensure a comfortable margin

This article reviews the critical optimization problem that can make this difference. This article will also present the different possible scenarios to optimize the significant costs of a company by proposing to opt for renewable energies.
A number of optimization tools will be discussed and analyzed in this article.

***Keywords*—** Renewable energy, optimization, supply chain

I. Introduction

Nowadays, Companies are carrying heavier and heavier costs than before, there are some who spend more money on marketing to market their products and achieve their objectives in Turnover, others prefer to invest in margin to have a competitive price compared to competitors ...
The methods to achieve the gain objective are known by almost all companies, in terms of product marketing for commercial companies, or to have a competitive cost price for production companies, or a cost of storage or low transportation for Logistics Company…[1]

on the other hand they all undergo very heavy loads which makes them lose all the gain which they had in their activities.

The purpose of this article is to propose solutions to optimize part of the business expenses: energy expenses.

According to the economist, energy consumption increased by +4.5 at the end of 2017, after + 1.9% at the end of 2016. According to the same newspaper, an increase of + 7.9% came from energy addressed to the national productive sector and + 3% concerning low voltage energy addressed mainly to households. In addition, the consumption of electricity went from 12,453 GWH to 37,446 GWH from 1998 to 2018. This increase reflects the dynamism of our country both economically and socially. And therefore; other solutions are needed to allow businesses to be more profitable.

II. Solar energy

Our kingdom is one of the sunny countries most of the year, even in winter, something that cannot be found in the most developed countries in Europe. Moreover, the construction of several solar power plants in the various regions of Morocco provides for the realization of additional solar capacity for the years to come.

Furthermore, solar energy has become the choice of a Moroccan population that can be considered important, but we don't see companies that opt for this solution to reduce the electricity bill when their consumption far exceeds that of houses.[2]
According to the economist: "*The Noor Midelt I project, awarded to the EDF Renouvelables, Masdar and Green of Africa consortium, should enable Ma¬roc to move from 3rd to 2nd place in the world CSP market This plant will have an installed capacity of 800 MW, almost the equivalent of that of a conventional nuclear reactor (1,000 MW). It will have to feed 1.19 million inhabitants and produce a kwh at 0.68 DH. For the period 2019-2023, the equipment*

plan provides for the realization of an additional solar capacity of 2,015 MW (120 Noor PV Tafilalt in 2019, 200 MW Noor PV Atlas in 2020, 200 MW in Koudia Baida in 2023, 300 MW to be carried out in 2023 under the law 13-09)"[3]

III. BIOMASS

On the other hand, all the companies that have organic waste can use this waste to proscribe energy, citing all the food, cardboard and paper companies, mass distribution like hypermarkets and supermarkets

Biomass, this energy source not yet exploited in Morocco, and which can save the costs of the energy company by exploiting its waste

2.1 definition of recycling

Waste recycling is the direct reintroduction of a waste into the production cycle from which it comes, this means that the waste is transformed into a raw material which will be used to produce new consumer goods while avoiding to draw resources from the planet.

These wastes can be used to produce energy, including methanization,
this technology based on degradation by microorganisms of organic matter, under controlled conditions and in the absence of oxygen, this degradation causes:

- Digestate : a moist product, rich in partially stabilized organic matter
- Biogaz: gas mixture saturated with water at the outlet of the digester and composed of approximately 50% to 70% methane (CH_4), 20% to 50% carbon dioxide (CO_2) and some trace gases (NH_3, N_2, H_2S)

2.2 advantages of recycling

this anaerobic digestion produces a double valuation of organic matter and energy, this is the specific interest in anaerobic digestion, compared to other sectors, decreases the amount of waste,
also allows a reduction in greenhouse gas emissions by replacing the use of fossil fuels or chemical fertilizers and can treat greasy or very wet organic waste. [4]

3.2 Agreed enterprise and biomass plant

The idea is to question the possibility that a business could benefit from energy from its own waste; have an agreement so that each company that gives its waste to a biomass power plant benefits from energy according to the weight of the waste [5]

REFERENCES

[1] A.A. Rentizelas*, I.P. Tatsiopoulos, A. Tolis " An optimization model for multi-biomass tri-generation energy supply,"

[2] Peter McKendry" Energy production from biomass (part 1): overview of biomass. "

[3] Weimin Wanga, Radu Zmeureanua,, HuguesRivard "Applying multi-objective genetic algorithmsin green building design optimization"

[4] Zhi ZhouJuan Manuel De BedoutJohn Michael KernEmrah BiyikRamu Sharat Chandra "Energy optimization system"

[5] DAI LIN "THE DEVELOPMENT AND PROSPECTIVE OF BIOENERGY TECHNOLOGY IN CHINA"

Existence of solutions for hybrid equation involving the Riemann -Liouville differential operators

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***Abstract*—In this paper, we study the existence of solutions for the following fractional hybrid differential equations involving Riemann-Liouville differential operators of order $1 < \alpha \leq 2$. An existence theorem for fractional hybrid differential equations is proved under mixed Lipschitz and Carathéodory conditions and using the Dhage point fixe theorem.**

***Index Terms*—Fractional, Riemann, Hybrid**

I. Introduction

During the past decades, fractional differential equations have attracted many authors [1], [4], [5], [7], [8], [9], [10], [11]. The differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature (for instance, to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive study during recent years.

The quadratic perturbations of nonlinear differential equations have attracted much attention. We call such fractional hybrid differential equations. There have been many works o n the theory of hybrid differential equations, and we refer the readers to the articles [2], [3], [4], [5], [6], [7].

Dhage and Lakshmikantham [3] discussed the following first order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad a.e \quad t \in J = [0,1], \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $f \in C^1(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in Car(J \times \mathbb{R}, \mathbb{R})$.
($Car(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions).

They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparaison results.

Zhao, Sun, Han and Li [11] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D_R^{\alpha}\Big[\dfrac{x(t)}{f(t,x(t))}\Big] = g(t,x(t)) \quad a.e \quad t \in J = [0,T], \\ x(0) = 0 \quad , \end{cases} \tag{2}$$

where $f \in C^1(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in Car(J \times \mathbb{R}, \mathbb{R})$.

The authors of [11] established the existence theorem for fractional hybrid differential equation and some fundamental differential inequalities. They also established the existence of extremal solutions.

Hilal and Kajouni [5] studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$

$$\begin{cases} D_C^{\alpha}\Big[\dfrac{x(t)}{f(t,x(t))}\Big] = g(t,x(t)) \quad a.e \quad t \in J = [0,T], \\ a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{3}$$

where $f \in C^1(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in Car(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparaison principle is proved which will be useful for further study of qualitative behavior of solutions.

In this paper we consider the fractional hybrid differential equations with involving Riemann -Liouville differential operators of order $1 < \alpha \leq 2$

$$\begin{cases} D_R^{\alpha}\Big[\dfrac{x(t)}{f(t,Bx(t))}\Big] = g(t,Bx(t)) \quad a.e \quad 0 \leq t < 1, \\ x(1) = x'(1) = 0, \end{cases} \tag{4}$$

where $f \in C^1(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in Car(J \times \mathbb{R}, \mathbb{R})$.
The term $Bx(t)$ is given by: $Bx(t) := \int_0^t K(t,s)x(s)ds$ where $K \in C(D, \mathbb{R}^+)$, the set of all positive functions which are continuous on $D := \{(t,s) \in \mathbb{R}^2/0 \leq s \leq t \leq T\}$ and

$$B^* = \sup_{t\in[0,1]} \int_0^t K(t,s)ds < \infty \tag{5}$$

Using the fixed point theorem, we give an existence theorem of solutions for the boundary value problem of the above nonlinear fractional differential equation under both Lipschitz and Carathéodory conditions. We present two examples to illustrate our results.

II. MOTIVATION & METHODOLOGY

A. Motivation

III. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\} \quad .$$

We denote by $Car(J \times \mathbb{R}, \mathbb{R})$ the class of functions $g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ such that
(i) the map $t \longmapsto g(t,x)$ is mesurable for each $x \in \mathbb{R}$ and
(ii) the map $x \longmapsto g(t,x)$ is is continuous for each$\in J$.
The class $Car(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J endowed with the norm $\| \, . \, \|_{L^1}$ defined by

$$\| y \|_{L^1} = \int_0^1 | y(s) | \, ds.$$

***Definition** 3.1:* [6]
The Riemann-Liouville fractional integral of the continuous function $h : (0, \infty) \longrightarrow \mathbb{R}$ of order $\alpha > 0$ is defined by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds$$

Provided that the right side is pointwise defined on $(0, \infty)$

***Definition** 3.2:* [6]
The Riemann-Liouville fractional derivative of order $\alpha > 0$ of the continuous function $h : (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$_0D_R^{\alpha}h(t) \;\; = \;\; \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1}h(s)ds, \tag{6}$$

where $n = [\alpha] + 1$, $[\alpha]$ denote the integer part of number α, Provided that the right side is pointwise defined on $(0, \infty)$.
From the definition of the Riemann-Liouville derivative, we can obtain the following statement

Lemma 3.1: [6]
Let $\alpha > 0$. If we assume $x \in C(0,1) \cap L(0,1)$, then the fractional differential equation

$$_RD_{0+}^{\alpha}x(t) = 0$$

has $x(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_nt^{\alpha-n}$, $c_i \in \mathbb{R}, i = 1, ..., n,$ as unique solutions, where n is the smallest integer greater than or equal to α.

Lemma 3.2: [6]
Assume $x \in C(0,1) \cap L(0,1)$ with a fractional derivative of $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}x(t) = x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_nt^{\alpha-n}$$

for some $c_i \in \mathbb{R}, \; i = 1, 2, ..., n$ where n is the smallest integer greater than or equal to α.

Lemma 3.3:
Let $h \in C[0,1]$ et $1 < \alpha \leq 2$. The unique solution of the problem

$$\begin{cases} D^{\alpha}\Big(\frac{x(t)}{f(t,Bx(t))}\Big) = h(t) \quad a.e \quad 0 \leq t < 1 \quad , \\ x(1) = x'(1) = 0, \end{cases} \tag{7}$$

is

$$x(t) = f(t,Bx(t)) \int_0^1 H(t,s)h(s)ds \quad , \tag{8}$$

where

$$H(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, 0 \leq s \leq t \leq 1 \\ \\ \frac{-t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \quad , \quad 0 \leq t \leq s \leq 1 \end{cases} \tag{9}$$

Preuve::
Applying the Riemann-Liouville fractional integral of the order α for the equation (7), we obtain

$$\frac{x(t)}{f(t,Bx(t))} = I^{\alpha}h(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2}$$

for some $c_1, c_2 \in \mathbb{R}$.
Consequently, the general solution of (7) is

$$x(t) = f(t,Bx(t))\Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}h(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}\Big) \quad . \tag{10}$$

By $x(1) = 0$ then

$$c_1 + c_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}h(s)ds.$$

From (10) we get

$$\frac{x'(t)f(t,Bx(t)) - x(t)f_t(t,Bx(t))}{f^2(t,Bx(t))} =$$

$$\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}h(s)ds + (\alpha-1)c_1t^{\alpha-2} + (\alpha-2)c_2t^{\alpha-3} \quad ,$$

by $x'(1) = 0$ we have

$$(\alpha-1)c_1 + (\alpha-2)c_2 = \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)ds.$$

Then

$$\begin{cases} c_1 = \frac{1}{\Gamma(\alpha-1)}\int_0^1((1-s)^{\alpha-2}-(1-s)^{\alpha-1})h(s)ds \\ c_2 = \frac{1}{\Gamma(\alpha-1)}\int_0^1((1-s)^{\alpha-1}-(1-s)^{\alpha-2})h(s)ds - \\ \frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}h(s)ds \end{cases}$$

therefore

$$\begin{aligned} x(t) &= f(t,Bx(t))\Big(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &+ \int_0^1 \Big(\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}((1-s)^{\alpha-1} \\ &- (1-s)^{\alpha-2}) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}((1-s)^{\alpha-1} \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha)}((1-s)^{\alpha-2}-(1-s)^{\alpha-1})\Big)h(s)ds\Big) \\ &= f(t,Bx(t))\Big(\int_0^t \Big(\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ &+ \frac{t^{\alpha-1}}{\Gamma(\alpha-1)}((1-s)^{\alpha-1}-(1-s)^{\alpha-2}) \\ &- \frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}((1-s)^{\alpha-2}-(1-s)^{\alpha-1})\Big)h(s)ds \\ &+ \int_t^1 \Big(\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}((1-s)^{\alpha-1}-(1-s)^{\alpha-2}) \\ &- \frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-1} \\ &+ \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}((1-s)^{\alpha-2}-(1-s)^{\alpha-1})\Big)h(s)ds\Big) \\ &= f(t,Bx(t))\Big(\int_0^t \Big(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &+ \frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &- \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)h(s)ds \\ &+ \int_t^1 \Big(\frac{s(1-t)t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &- \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)h(s)ds\Big) \\ &= f(t,Bx(t))\int_0^1 H(t,s)h(s)ds \quad , \end{aligned}$$

The proof is complete. ■

Lemma 3.4:
The function $H(t,s)$ defined by (9) satisfies the following conditions

$$\Gamma(\alpha-1)H(t,s) \leq \; q(t)k(s) \quad , \qquad (11)$$

where $q(t) = (1-t)t^{\alpha-2}$ and $k(s) = s(1-s)^{\alpha-2}$.

IV. Existence result

In this section, we prove the existence results for the hybrid differential equations with fractional order (4) on the closed and bounded interval $J = [0,1]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it.
We defined the multiplication in X by $(xy)(t) = x(t)y(t)$ for $x, y \in X$.
Clearly $X = C(J,\mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it.

Lemma 4.1: [2]
Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A_1 : X \longrightarrow X$ and $A_2 : X \longrightarrow X$ be two operators such that

(a) A_1 is Lipschitzian with a Lipschitz constant L
(b) B is completely continuous,
(c) $x = A_1xA_2y \Longrightarrow x \in S$ for all $y \in S$, and
(d) $LM < 1$, where $M = \|A_2(S)\| = sup\{\|A_2(x)\| : x \in S\}$

then the operator equation $x = A_1xA_2y$ has a solution in S
We make the following assumptions
(H_0) The function $x \longmapsto \frac{x}{f(t,Bx)}$ is increasing in $\mathbb{R}$ almost every where for $t \in J$.
(H_1) There exists a constant $L > 0$ such that

$$| f(t,Bx) - f(t,By) | \leq LB^*|x-y| = L^*|x-y| \quad ,$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $L^* = LB^*$.
(H_2) There exists a function $h \in L^1(J,\mathbb{R}^+)$ such that

$$|g(t,Bx)| \leq B^*h(t) \quad a.e \quad t \in J \quad ,$$

for all $x \in \mathbb{R}$.
For convenience we denote

$$T = \frac{1}{\Gamma(\alpha-1)}\int_0^1 k(s)ds \quad . \qquad (12)$$

Theorem 4.1: Assume that hypotheses (H_1) and (H_2) hold. Further, if

$$L^*B^*T\|h\|_{L^1} < 1, \qquad (13)$$

then the boundary value problem (4) has a solution define J.

***Preuve:*:**
We define a subset S of X by

$$S = \{x \in X/\|x\| \leq N\} \quad ,$$

where

$$N = \frac{B^*F_0T\|h\|_{L^1}}{1-B^*L^*T\|h\|_{L^1}},$$

et

$$F_0 = \sup_{t\in J}|f(t,0)| \quad .$$

It is clear that S satisfies hypothesis of lemma 4.1.
By application of Lemma 4.1, the equation (4) is equivalent to the nonlinear hybrid integral equation

$$x(t) = f(t, Bx(t)) \int_0^1 H(t,s) g(s, Bx(s)) ds \quad , \quad t \in J. \tag{14}$$

Define two operators $A_1 : X \longrightarrow X$ and $A_2 : S \longrightarrow X$ by

$$A_1 x(t) = f(t, Bx(t)), \quad t \in J \tag{15}$$

and

$$A_2 x(t) = \int_0^1 H(t,s) g(s, Bx(s)) ds \quad . \tag{16}$$

Then the hybrid integral equation (14) is transformed into the operator equation as

$$x(t) = A_1 x(t) A_2 x(t) \quad , \quad t \in J \quad . \tag{17}$$

We shall show that the operators A_1 and A_2 satisfy all the conditions of Lemma 4.1.
Claim 1, Let $x, y \in X$ then by hypothesis (H_1),

$$\begin{aligned} |A_1 x(t) - A_1 y(t)| &= |f(t, Bx(t)) - f(t, By(t))| \\ &\le L^* |x(t) - y(t)| \\ &\le L^* \|x - y\| \quad , \end{aligned}$$

for all $t \in J$.
Taking supremum over t, we obtain t

$$\|A_1 x - A_1 y\| \le L^* \|x - y\| \quad ,$$

for all $x, y \in X$.
Claim 2, A_2 is a continuous in S.
Let (x_n) be a sequence in S converging to a point $x \in S$. and Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} A_2 x_n(t) &= \lim_{n\to\infty} \int_0^1 H(t,s) g(s, Bx_n(s)) ds \\ &= \int_0^1 H(t,s) \lim_{n\to\infty} g(s, Bx_n(s)) ds \\ &= \int_0^1 H(t,s) g(s, Bx(s)) ds \\ &= A_2 x(t) \quad , \end{aligned}$$

for all $t \in J$.
This shows that A_2 is a continuous operator on S.
Claim 3, A_2 is compact operator on S .
First, we show that $A_2(S)$ is a uniformly bounded set in X. Let $x \in S$ be arbitrary. By Lemma 3.4, we have

$$\begin{aligned} |A_2 x(t)| &= |\int_0^1 H(t,s) g(s, Bx(s)) ds| \\ &\le q(t) \frac{1}{\Gamma(\alpha - 1)} B^* \int_0^1 k(s) h(s) ds \\ &\le TB^* \|h\|_{L^1} \quad , \end{aligned}$$

for all $t \in J$.
Takin to sup from t, we obtain

$$\|A_2 x\| \le TB^* \|h\|_{L^1} \quad ,$$

for all $x \in S$.
so A_2 is uniformly bounded on S.
Next, we prove that $A_2(S)$ is an equi-continuous set on X. Given $\varepsilon > 0$ and let

$$\delta < \min\left\{\frac{1}{2}, \frac{\Gamma(\alpha+1)\varepsilon}{12\|h\|_{L^1}}\right\}$$

Let $x \in S$ et $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, $0 < t_2 - t_1 < \delta$.
We have

$$\begin{aligned} |A_2 x(t_2) - A_2 x(t_1)| &= \Big| \int_0^1 H(t_2, s) g(s, Bx(s)) ds \\ &\quad - \int_0^1 H(t_1, s) g(s, Bx(s)) ds \Big| \\ &\le B^* \|h\|_{L^1} \Big| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1} - t_2^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad + \int_0^{t_2} \frac{s(1-t_2) t_2^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &\quad - \int_{t_2}^1 \frac{t_2^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad + \int_{t_2}^1 \frac{s(1-t_2) t_2^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &\quad - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1} - t_1^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad - \int_0^{t_1} \frac{s(1-t_1) t_1^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &\quad + \int_{t_1}^1 \frac{t_1^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad - \int_{t_1}^1 \frac{s(1-t_1) t_1^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big| \end{aligned}$$

then

$$\begin{aligned} &|A_2 x(t_2) - A_2 x(t_1)| \le \\ &B^* \|h\|_{L^1} \Big(\int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &- \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds + (t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &+ (t_2^{\alpha-2} - t_1^{\alpha-2}) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &+ (t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \Big) \end{aligned}$$

$$\leq B^*\|h\|_{L^1}\Big(\frac{t_2^\alpha - t_1^\alpha + t_2^{\alpha-1} - t_1^{\alpha-1}}{\Gamma(\alpha+1)} + \frac{t_2^{\alpha-2} - t_1^{\alpha-2}}{\Gamma(\alpha)} + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\Gamma(\alpha)}\Big)$$

$$\leq \frac{B^*\|h\|_{L^1}}{\Gamma(\alpha+1)}\Big(t_2^\alpha - t_1^\alpha + (1+\alpha)(t_2^{\alpha-1} - t_1^{\alpha-1}) + \alpha(t_2^{\alpha-2} - t_1^{\alpha-2})\Big)$$

$$\leq \frac{B^*\|h\|_{L^1}}{\Gamma(\alpha+1)}\Big(t_2^\alpha - t_1^\alpha + 3(t_2^{\alpha-1} - t_1^{\alpha-1}) + 2(t_2^{\alpha-2} - t_1^{\alpha-2})\Big).$$

In order to estimate $t_2^\alpha - t_1^\alpha$, $t_2^{\alpha-1} - t_1^{\alpha-1}$ and $t_2^{\alpha-2} - t_1^{\alpha-2}$,
we consider the following cases

Case 1: $0 \leq t_1 < \delta$, $t_2 < 2\delta$.
$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < (2\delta)^\alpha \leq 2^\alpha\delta \leq 4\delta$,
$t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} < (2\delta)^{\alpha-1} \leq 2^{\alpha-1}\delta \leq 2\delta$
$t_2^{\alpha-2} - t_1^{\alpha-2} \leq t_2^{\alpha-2} < (2\delta)^{\alpha-2} \leq 2^\alpha\delta \leq \delta$

Case 2: $0 < t_1 < t_2 \leq \delta$.
$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < \delta^\alpha \leq \alpha\delta \leq 4\delta$, $t_2^{\alpha-1} - t_1^{\alpha-1} \leq t_2^{\alpha-1} < \delta^{\alpha-1} \leq (\alpha-1)\delta \leq 2\delta$
$t_2^{\alpha-2} - t_1^{\alpha-2} \leq t_2^{\alpha-2} < \delta^{\alpha-2} \leq (\alpha-2)\delta \leq \delta$

Case 3: $\delta \leq t_1 < t_2 \leq 1$.
$t_2^\alpha - t_1^\alpha \leq \alpha\delta \leq 4\delta$, $t_2^{\alpha-1} - t_1^{\alpha-1} \leq (\alpha-1)\delta < 2\delta$
$t_2^{\alpha-2} - t_1^{\alpha-2} \leq (\alpha-2)\delta < \delta$

we obtain

$$|A_2x(t_2) - A_2x(t_1)| < \varepsilon \quad ,$$

for all $t_1, t_2 \in J$ and all $x \in X$.
This implies that $A_2(S)$ is an equi-continuous set in X.
Then by Arzelà-Ascoli theorem, A_2 is a continuous and compact operator on S.
Claim 4, The hypothesis (c) of lemma 4.1 is satisfied.
Let $x, y \in X$ such that $x = A_1xA_2y$. Then

$$\begin{aligned} |x(t)| &= |A_1x(t)||A_2y(t)| \\ &= |f(t, Bx(t)) - f(t,0) \\ &+ f(t,0)||\int_0^1 H(t,s)g(s, Bx(s))ds| \\ &\leq B^*[L^*|x(t)| + F_0]\Big(q(t)\frac{1}{\Gamma(\alpha-1)}\int_0^1 k(s)h(s)ds\Big) \\ &\leq B^*[L^*|x(t)| + F_0]T\|h\|_{L^1} \quad . \end{aligned}$$

Thus,

$$|x(t)| \leq \frac{B^*F_0T\|h\|_{L^1}}{1 - B^*L^*T\|h\|_{L^1}},$$

Taking supremum over t,

$$\|x\| \leq \frac{B^*F_0T\|h\|_{L^1}}{1 - B^*L^*T\|h\|_{L^1}}.$$

Then $x \in S$ and the hypothesis (c) of Lemma 4.1 is satisfied.
Finally, we have
$M = \|A_2(S)\| = \sup\{\|A_2x\| : x \in S\} \leq B^*T\|h\|_{L^1}$,
so,

$$L^*M \leq L^*B^*T\|h\|_{L^1} < 1 \quad .$$

Thus, all the conditions of Lemma 4.1 are satisfied.
Hence the operator equation $A_1xA_2x = x$ has a solution in S. As a result, the boundary value problem (4) has a solution defined on J. This completes the proof. ■

V. EXEMPLES

In this section, we will present two examples to illustrate the main results.

A. Exemple 1

we consider the fractional hybrid differential equation

$$\begin{cases} D^{\frac{3}{2}}x(t) = \sin x \quad p.p. \quad 0 \leq t < 1 \quad , \\ x(1) = x'(1) = 0 \quad , \end{cases} \tag{18}$$

whetre $f(t,x) = 1$, $g(t,x) = \sin x$ and $h(t) = 1$.
Then hypothesis (H_1) and (H_2) hold.
Since

$$\begin{aligned} T &= \frac{1}{\Gamma(\alpha-1)}\int_0^1 k(s)ds \\ &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 s(1-s)^{\frac{1}{2}}ds \\ &= \frac{4}{35\sqrt{\pi}} \quad , \end{aligned}$$

choosing $L = 1$, then we have

$$LT\|h\|_{L^1} < 1 \quad .$$

Therefore, the fractional hybrid differential equation (18) has a solution.

B. Exemple 2

we consider the fractional hybrid differential equation

$$\begin{cases} D^{\frac{3}{2}}\left[\frac{x(t)}{\sin x + 2}\right] = \cos x \quad p.p. \quad 0 \leq t < 1 \\ x(1) = x'(1) = 0 \end{cases} \tag{19}$$

where $f(t,x) = \sin x + 2$, $g(t,x) = \cos x$ et $h(t) = 1$.
Then hypothesis (H_1) and (H_2) hold.
Since

$$T = \frac{4}{35\sqrt{\pi}} \quad .$$

choosing $L = 1$, then

$$LT\|h\|_{L^1} < 1 \quad .$$

Therefore, the fractional hybrid differential equation (19) has a solution on $[0, 1]$.

References

[1] R.P. Agarwal. A propos d'une note de M. Pierre Humbert. C. R. Académie des Sciences 236, 2031-2032 (1953).
[2] B.C. Dhage, On a fixed point theorem in Banach algebras with applications, Appl. Math. Lett. 18 (2005) 273-280.
[3] B.C. Dhage, V. Lakshmikantham, Basic results on hybrid differential equations, Nonlinear Anal. Hybrid 4 (2010) 414-424.
[4] B.C. Dhage, V. Lakshmikantham, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, Diff. Eq. et App. 2 (2010) 465-486
[5] K. Hilal, A. Kajouni, Boundary value problems for hybrid differential equations with fractional order, Adv. Differ. Equ. (2015) 2015:183
[6] A. A. Kilbas, H. H. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science B.V, Amsterdam, 2006.
[7] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. TMA 69 (2008) 2677-2682.
[8] V. Lakshmikantham, J.V. Devi, Theory of fractional differential equations in Banach Space, Eur. J. Pure Appl. Math. 1 (2008) 38-45.
[9] K. B. Oldham, J. Spanier,The fractional calculus, Academic Press, New York, 1974.
[10] I. Podlubny, Fractional Differential Equations, Mathematics in Sciences and Engineering,198, Academic Press, San Diego, 1999.
[11] S. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equations, Electronic Journal of Differential Equations,vol. 36, pp. 1-12, 2006.
[12] Y. Zhao, S. Suna, Z. Han, Q. Li, Theory of fractional hybrid differential equations, Computers and Mathematics with Appl. 62 (2011) 1312-1324

On the Cauchy problem for the fractional drift-diffusion system in critical Fourier-Besov-Morrey spaces

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Abstract—**In this paper, we study the Cauchy problem of the fractional drift-diffusion system. By using the Fourier localization argument and the Littlewood Paley theory, we get the local well-posedness for large initial data in critical Fourier-Besov-Morrey space $\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}$, Moreover, if the initial data is sufficiently small then the solution is global.**

Index Terms—**Drift-diffusion, Local existence, Littlewood-Paley theory, Fourier-Besov-Morrey spaces.**

I. Introduction

In this paper, we consider the following Cauchy problem for the fractional drift-diffusion system in $\mathbb{R}^n \times \mathbb{R}^+$ with fractional Laplacian

$$\begin{cases} \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = -\nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \partial_t w + (-\Delta)^{\frac{\alpha}{2}} w = \nabla \cdot (w \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v - w & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the unknown functions $v = v(x,t)$ and $w = w(x,t)$ denote densities of the electron and the hole in electrolytes, respectively, $\phi = \phi(x,t)$ denotes the electric potential, $v_0(x)$ and $w_0(x)$ are initial datum. Throughout this paper, we assume that $n \geq 2$ and $1 < \alpha \leq 2$.
Notice that the function ϕ is determined by the Poison equation in the third equation of (1), and it's given by:

$$\phi(x,t) = (-\Delta)^{-1}(w - v)(x,t).$$

So that the system (1) can be rewritten as the following system:

$$\begin{cases} \partial_t v + (-\Delta)^{\frac{\alpha}{2}} v = -\nabla \cdot \left(v \nabla (-\Delta)^{-1}(w - v)\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ \partial_t w + (-\Delta)^{\frac{\alpha}{2}} w = \nabla \cdot \left(w \nabla (-\Delta)^{-1}(w - v)\right) & \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) = v_0(x), \quad w(x,0) = w_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2}$$

Mathematical analysis of the Drift-diffusion system has drawn much attention during the past three decades, we refer the reader to see [1], [5] and the references therein for previous works on this system concerning existence of classical solutions and weak solutions.

In the context of Besov spaces and for $\alpha = 2$, Karch in [14] proved existence of global solution of the system (1) with small initial data in critical Besov space $\dot{\mathcal{B}}_{p,\infty}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $\frac{n}{2} \leq p < n$. After, Deng and Li [9] showed that the system (1) is well-posed in $\dot{\mathcal{B}}_{4,2}^{-\frac{3}{2}}(\mathbb{R}^2)$, and ill-posed in $\dot{\mathcal{B}}_{4,r}^{-\frac{3}{2}}(\mathbb{R}^2)$ for $2 < r \leq \infty$. Zhao, Liu, and Cui [21] established the existence of global and local solution of the system (1) in critical Besov space $\dot{\mathcal{B}}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ with $1 < p < 2n$ and $1 \leq r \leq \infty$.

We mention here that if w vanishes ($w = 0$) and for $\alpha = 2$, the system (1) becomes to the well-known Keller-Segel model of chemotaxis:

$$\begin{cases} \partial_t v = \Delta v - \nabla \cdot (v \nabla \phi) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \Delta \phi = v & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{3}$$

In the paper [4] the local well-posedness of the system (3) has been proved in the three-dimensional case. Iwabuchi and Nakamura [12], [13] get the global well-posednes of (3) for small initial data in the critical space

$$\dot{\mathcal{B}}_{p,r}^{-2+\frac{n}{p}}(\mathbb{R}^n)$$

with $1 \leq p < \infty$ and $1 \leq r \leq \infty$ Inspired by the work [21], The purpose of this paper is to establish the existence of local solution to (1) for large initial data and global solution for small initial data in the critical Fourrier-Besov-Morrey space

$$\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}$$

Let us firstly recall the scaling property of the systems: if (v, w) solves (1) with initial data (v_0, w_0) (ϕ can be determined by (v, w)), then (v_γ, w_γ) with $(v_\gamma(x,t), w_\gamma(x,t)) := (\gamma^\alpha v(\gamma x, \gamma^\alpha t), \gamma^\alpha w(\gamma x, \gamma^\alpha t))$ is also a solution to (1) with the initial data

$$(v_{0,\gamma}(x), w_{0,\gamma}(x)) := (\gamma^\alpha v_0(\gamma x), \gamma^\alpha w_0(\gamma x)) \tag{4}$$

(ϕ_γ can be determined by (v_γ, w_γ)).

Definition 1.1: A critical space for initial data of the system (1) is any Banach space $E \subset \mathcal{S}'(\mathbb{R}^n)$ whose norm is invariant under the scaling (4) for all $\gamma > 0$, i.e

$$\|(v_{0,\gamma}(x), w_{0,\gamma}(x))\|_E \approx \|(v_0(x), w_0(x))\|_E.$$

Under these scalings, We can show that the space pair $\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \times \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}$ is critical for (1) see (Remark 2.1 for details).

In order to solve the equation (1), we consider the following equivalent integral system

$$\begin{cases} v(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0 - \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (v \nabla \phi(\tau))\, d\tau \\ w(t) = e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_0 + \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla \cdot (w \nabla \phi(\tau))\, d\tau. \end{cases} \tag{5}$$

With $\mathcal{F}\left((-\Delta)^{\frac{\alpha}{2}} f\right)(\xi) = |\xi|^\alpha \mathcal{F} f(\xi)$.

Throughout this paper, we use $\mathcal{FN}^s_{p,\lambda,q}$ to denote the homogenous Fourier Besov-Morrey spaces, $(v,w) \in X$ to denote $(v,w) \in X \times X$ for a Banach space X $\Big($the product $X \times X$ will be endowed with the usual norm $\|(v,w)\|_{X\times X} := \|v\|_X + \|w\|_X$ $\Big)$, $\|(v,w)\|_X$ to denote $\|(v,w)\|_{X\times X}$, $V \lesssim W$ means that there exists a constant $C > 0$ such that $V \leq CW$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

Now we present our main results as follows.

Theorem 1.1: Let $n \geq 2$, $1 < \alpha \leq 2$, $\rho_0 > \frac{\alpha}{\alpha-1}$, $\max\{n-(n+3-2\alpha)p, 0\} \leq \lambda < n$, $1 \leq p < \infty$, $q \in [1,\infty]$, $(v_0,w_0) \in \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ *and* $\frac{1}{\rho_0} + \frac{1}{\rho_0'} = 1$.

Then there exists $T \geq 0$ such that the system (1) has a unique local solution
$(v,w) \in X_T$, where

$$X_T = \mathfrak{L}^{\rho_0}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$$

and

$$(v,w) \in \mathcal{C}\left(0,T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right).$$

Besides, there exixts $K \geq 0$ such that if (v_0,w_0) satisfies: $\|(v_0,w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \leq K$, then the above assertion holds for $T = \infty$; i.e, the solution (v,w) is global.

II. Preliminaries

In this section, we give some notations and recall basic properties about Fourier-Besov-Morrey spaces that will be used throughout the paper.

The Fourier-Besov-Morrey spaces were introduced in [10] are constructed via a type of localization on Morrey spaces.

We define the function spaces M^λ_p.

Definition 2.1: [15] Let $1 \leq p \leq \infty$ and $0 \leq \lambda < n$. The homogeneous Morrey space M^λ_p is the set of all functions $f \in L^p(B(x_0,r))$ such that

$$\|f\|_{M^\lambda_p} = \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty, \tag{6}$$

where $B(x_0,r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r > 0$.

The space M^λ_p endowed with the norm $\|f\|_{\mathrm{M}^\lambda_p}$ is a Banach space.

When $p = 1$, the L^1 -norm in (6) is understood as the total variation of the measure f on $B(x_0,r)$ and M^λ_p as a subspace of Radon measures. When $\lambda = 0$, we have $M^0_p = L^p$.

The proofs of the results presented in this paper are based on a dyadic partition of unity in the Fourier variables, the so-called, homogeneous Littlewood-Paley decomposition. We recall briefly this construction below. For more detail, we refer the reader to [2].

Let $f \in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$$

and its inverse Fourier transform as

$$\check{f}(x) = \mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

Let $\varphi \in S(\mathbb{R}^d)$ be such that $0 \leq \varphi \leq 1$ and $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi)$$

and

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x).$$

We now present some frequency localization operators:

$$\dot{\Delta}_j f = \varphi_j(D)f = 2^{dj} \int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)dy$$

and

$$\dot{S}_j f = \sum_{k\leq j-1} \dot{\Delta}_k f = \psi_j(D)f = 2^{dj} \int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y)dy.$$

From the definition, one easily derives that

$$\dot{\Delta}_j \dot{\Delta}_k f = 0, \quad \text{if } |j-k| \geq 2$$
$$\dot{\Delta}_j\left(\dot{S}_{k-1} f \dot{\Delta}_k f\right) = 0, \quad \text{if } |j-k| \geq 5.$$

The following Bony paraproduct decomposition will be applied throughout the paper.

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v)$$

where $\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v$, $\tilde{\Delta}_j v = \sum_{|j'-j|\leq 1} \dot{\Delta}_{j'} v$.

Lemma 2.1: [10] Let $1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \lambda_1, \lambda_2, \lambda_3 < n$.

(i) (Hölder's inequality) Let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathrm{M}^{\lambda_3}_{p_3}} \leq \|f\|_{\mathrm{M}^{\lambda_1}_{p_1}} \|g\|_{\mathrm{M}^{\lambda_2}_{p_2}}. \tag{7}$$

(ii) (Young's inequality) If $\varphi \in L^1$ and $g \in \mathrm{M}^{\lambda_1}_{p_1}$, then

$$\|\varphi * g\|_{\mathrm{M}^{\lambda_1}_{p_1}} \leq \|\varphi\|_{L^1} \|g\|_{\mathrm{M}^{\lambda_1}_{p_1}}, \tag{8}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.2: [10] Let $1 \leq q \leq p < \infty$, $0 \leq \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \leq \frac{n-\lambda_2}{q}$ and let γ be a multi-index. If $supp(\widehat{f}) \subset \{|\xi| \leq A2^j\}$, then there is a constant $C > 0$ independent of f and j such that

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathrm{M}_q^{\lambda_2}} \leq C 2^{j|\gamma| + j(\frac{n-\lambda_2}{q} - \frac{n-\lambda_1}{p})} \|\widehat{f}\|_{\mathrm{M}_p^{\lambda_1}}. \tag{9}$$

Then, we define the function spaces $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$, see [10].

Definition 2.2: (Homogeneous Fourier-Besov-Morrey spaces)
Let $1 \leq p, q \leq \infty$, $0 \leq \lambda < n$ and $s \in \mathbb{R}$. The Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f \in \mathcal{S}' \backslash \mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that $\varphi_j \hat{f} \in \mathrm{M}^\lambda_p$, for all $j \in \mathbb{Z}$, and

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsq} \left\|\varphi_j \hat{f}\right\|^q_{\mathrm{M}^\lambda_p} \right)^{\frac{1}{q}} & for \quad q < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\|\varphi_j \hat{f}\right\|_{\mathrm{M}^\lambda_p} & for \quad q = \infty. \end{cases} \tag{10}$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (10) is a Banach space. Since $\mathrm{M}^0_p = L^p$, we have $\mathcal{FN}^s_{p,0,q} = FB^s_{p,q}$,$\mathcal{FN}^s_{1,0,q} = FB^s_{1,q} = \mathcal{B}^s_q$ and $\mathcal{FN}^{-1}_{1,0,1} = \chi^{-1}$ where $\mathcal{B}^s_q$ is the Fourier-Herz space and χ^{-1} is the Lei-Lin space [18].

Remark 2.1: The space pair $\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \times \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ is critical for (1). For this, set $u_{0,\gamma}(\xi) = \gamma^{2-2\alpha} u_0(\gamma\xi)$, then its Fourier transform is $\widehat{u_{0,\gamma}}(\xi) = \gamma^{2-2\alpha-n} \hat{u}_0\left(\gamma^{-1}\xi\right)$.
Let

$$\begin{aligned} f_j(\xi) &\stackrel{\text{def}}{=} \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right) \widehat{u_{0,\gamma}}(\xi) \\ &= \varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right) \gamma^{2-2\alpha-n} \hat{u}_0\left(\gamma^{-1}\xi\right) \end{aligned}$$

By change of variable, we get

$$\begin{aligned} &\|f_j\|_{\mathrm{M}^\lambda_p} \\ &= \gamma^{2-2\alpha-n} \left\|\varphi\left(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi\right) \hat{u}_0\left(\gamma^{-1}\xi\right)\right\|_{\mathrm{M}^\lambda_p} \\ &= \gamma^{2-2\alpha-n} \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \\ &\left\|\varphi\left(2^{-j+[\log_2\gamma]}\gamma^{-1}\xi\right) \widehat{u_0}\left(\gamma^{-1}\xi\right)\right\|_{L^p(B(x_0,r))} \\ &= \gamma^{2-2\alpha-n}\gamma^{\frac{n}{p}}\gamma^{\frac{-\lambda}{p}} \sup_{x_0\in\mathbb{R}^n} \sup_{r>0} \left(\gamma^{-1}r\right)^{-\frac{\lambda}{p}} \\ &\left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right) \widehat{u_0}(\eta)\right\|_{L^p(B(\gamma^{-1}x_0,\gamma^{-1}r))} \\ &= 2^{\left(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}\right)\log_2\gamma} \left\|\varphi\left(2^{-j+[\log_2\gamma]}\eta\right) \widehat{u_0}(\eta)\right\|_{\mathrm{M}^\lambda_p}, \end{aligned}$$

which implies

$$\begin{aligned} &\|\{2^{j(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p})}\|f_j(\xi)\|_{M^\lambda_p}\}\|_{l^q} \\ &= \|\{2^{j(2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p})} 2^{\log_2\gamma(2\alpha-2-\frac{n}{p'}+\frac{\lambda}{p})} \|\varphi_{j-[\log_2\gamma]}\widehat{u_0}(\xi)\|_{M^\lambda_p}\}\|_{l^q} \\ &\approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}}_{p,\lambda,q}} \end{aligned}$$

and since

$$\varphi_j(\xi)\widehat{u_{0,\gamma}}(\xi) = \sum_{|k-j|\leq 2} \varphi_j(\xi) f_k(\xi),$$

we can get

$$\|u_{0,\gamma}\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}-\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|u_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Similary, we have

$$\|w_{0,\gamma}\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|w_0\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.$$

Now, we give the definition of the mixed space-time spaces.

Definition 2.3: Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t,x)\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}^\lambda_p)} \Big\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I, \mathcal{FN}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R} \times \mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^\rho(I,\mathcal{FN}^s_{p,\lambda,q})}$ norm.
According to Minkowski inequality, it is easy to verify that

$$\begin{aligned} L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho \leq q, \\ \mathcal{L}^\rho\left(I, \mathcal{FN}^s_{p,\lambda,q}\right) &\hookrightarrow L^\rho\left(I; \mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho \geq q, \end{aligned}$$

where $\|u(t,x)\|_{L^\rho}\left(I; \mathcal{FN}^s_{p,\lambda,q}\right) := \left(\int_I \|u(\tau,\cdot)\|^\rho_{\mathcal{FN}^s_{p,\lambda,q}} d\tau\right)^{1/\rho}$.

At the end of this section we recall an existence and uniqueness result for an abstract operator equation in a Banach space, which will be used to prove Theorem 1.1 in the sequel. For the proof, we refer the reader to see [17] and [3].

Lemma 2.3: Let X be a Banach space with norm $\|.\|_X$ and $B : X \times X \longmapsto X$ be a bounded bilinear operator satisfying

$$\|B(u,v)\|_X \leq \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ such that $\|y\|_X \leq \varepsilon$, the equation $x := y + B(x, x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \leq 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \leq 2\varepsilon$, then

$$\|\overline{x} - x'\|_X \leq \frac{1}{1-4\varepsilon\eta} \|y - y'\|_X.$$

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III. Linear estimates in Fourier-Besov-Morrey spaces

In this section, we will establish some crucial estimates in the proof of Theorem 1.1. We now consider the following linear estimates for the fractional heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$.

Lemma 3.1: Let I=(0, T), s $\in \mathbb{R}$, p, q, $\rho \in [1,\infty]$ and $0 \leq \lambda < n$. There exists a constant $C>0$ such that

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\|_{\mathcal{L}^\rho\left([0,T),\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{p,\lambda,q}\right)} \leq C\|u_0\|_{\mathcal{FN}^s_{p,\lambda,q}}, \quad (11)$$

where $u_0 \in \mathcal{FN}^s_{p,\lambda,q}$.

proof Since supp$\varphi_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we obtain

$$\begin{aligned}\left\|\mathcal{F}\left[\Delta_j e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\right]\right\|_{M_p^\lambda} &= \left\|\varphi_j e^{-t|\xi|^\alpha}\widehat{u_0}\right\|_{M_p^\lambda}\\ &\leq e^{-t2^{\alpha(j-1)}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}.\end{aligned}$$

Then, by the Minkowski inequality, we have

$$\begin{aligned}&\left\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}u_0\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{\alpha}{\rho}}_{p,\lambda,q}\right)}\\ &\leq \left\|\left\{2^{j\left(s+\frac{\alpha}{\rho}\right)}\left(\int_0^T e^{-t\rho 2^{\alpha(j-1)}}dt\right)^{\frac{1}{\rho}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}\right\}\right\|_{\ell^q}\\ &\leq \left\|\left\{2^{j\left(s+\frac{\alpha}{\rho}\right)}\left(\frac{1-e^{-T\rho 2^{(j-1)\alpha}}}{\rho 2^{\alpha(j-1)}}\right)^{\frac{1}{\rho}}\left\|\varphi_j\widehat{u}_0\right\|_{M_p^\lambda}\right\}\right\|_{\ell^q}\\ &\leq C\left\|u_0\right\|_{\mathcal{FN}^s_{p,\lambda,q}}.\end{aligned}$$

Lemma 3.2: [8] Let I=(0, T), s $\in \mathbb{R}$, p, q, $\rho \in [1,\infty]$ $0 \leq \lambda < n$ and $1 \leq r \leq \rho$.
There exists a constant $C>0$ such that

$$\left\|\int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau\right\|_{\mathcal{L}^\rho\left(I;\mathcal{FN}^{s+\frac{2}{\rho}}_{p,\lambda,q}\right)} \leq C\|f\|_{\mathcal{L}^r\left(I;\mathcal{FN}^{s-2+\frac{2}{r}}_{p,\lambda,q}\right)}. \quad (12)$$

IV. Bilinear estimates in Fourier-Besov-Morrey spaces

Lemma 4.1: Let $I=(0,T)$, $s \in \mathbb{R}$, p, q $\in [1,\infty]$, $\max\{n-(n+3-2\alpha)p,0\} < \lambda < n$, $\rho_0 > \dfrac{\alpha}{\alpha-1}$ and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. There exists a constant $C>0$ such that

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^1\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \leq C\Bigg[&\|f\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)}\\ &\times\|g\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\\ +\|g\|_{\mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)} &\times\|f\|_{\mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)}\Bigg]\end{aligned}$$

for all $f \in \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right)$
and $g \in \mathfrak{L}^{\rho_0'}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}}_{p,\lambda,q}\right) \cap \mathfrak{L}^{\rho_0}\left(I;\mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}}_{p,\lambda,q}\right)$

Proof Applying Bony paraproduct decomposition and quasi-orthogonality property for Littlewood-Paley decomposition, for fixed j, we obtain

$$\begin{aligned}\dot{\Delta}_j(f\nabla g) &= \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g) + \sum_{|k-j|\leq 4}\dot{\Delta}_j(\dot{S}_{k-1}g\dot{\Delta}_k\nabla f)\\ &+ \sum_{k\geq j-3}\dot{\Delta}_j(\dot{\Delta}_k f\tilde{\dot{\Delta}}_k\nabla g)\\ &= I_j^1 + I_j^2 + I_j^3\end{aligned}$$

Then, by the triangle inequalities in M_p^λ and in $l^q(\mathbb{Z})$, we have

$$\begin{aligned}\|\nabla.(f\nabla g)\|_{\mathfrak{L}^1\left(I;\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} &\leq \|f\nabla g\|_{\mathfrak{L}^1\left(I;\mathcal{FN}^{3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)}\\ &\leq \{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{\dot{\Delta}_j(f\nabla g)}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &\leq \{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^1}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &+\{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^2}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &+\{\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\widehat{I_j^3}\|^q_{L^1(I,M_p^\lambda)}\}^{\frac{1}{q}}\\ &:= J_1+J_2+J_3\end{aligned}$$

By using the Young's inequality in Morrey spaces and Bernstein-type inequality with $|\gamma|=0$, we have

$$\left\|\varphi_j\hat{f}\right\|_{L^1} \leq C2^{j(\frac{n}{p'}+\frac{\lambda}{p})}\left\|\varphi_j\hat{f}\right\|_{M_p^\lambda}$$

Then

$$
\begin{aligned}
\|\widehat{I_j^1}\|_{L^1(I,M_p^\lambda)} &\le \sum_{|k-j|\le 4} \|\widehat{(\dot{S}_{k-1}f\dot{\Delta}_k\nabla g)}\|_{L^1(I,M_p^\lambda)}\\
&\le \sum_{|k-j|\le 4}\|\varphi_j\widehat{\nabla g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{l\le k-2}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,L^1)}\\
&\le C\sum_{|k-j|\le 4}2^k\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{l\le k-2}2^{(\frac{n}{p'}+\frac{\lambda}{p})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)}\\
&\le C\sum_{|k-j|\le 4}2^k\|\varphi_j\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{l\le k-2}2^{(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})l}\,2^{(2\alpha-2-\frac{\alpha}{\rho_0})l}\|\varphi_l\widehat{f}\|_{L^{\rho_0}(I,M_p^\lambda)}\\
&\le C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\sum_{|k-j|\le 4}2^k\Big(\sum_{l\le k-2}2^{l(2\alpha-2-\frac{\alpha}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\\
&\le C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\sum_{|k-j|\le 4}2^{k(2\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)},
\end{aligned}
$$

where we have used the fact that $\rho_0 > \dfrac{\alpha}{\alpha-1}$ in the last inequality.
Thus, by using the Young inequality, we have

$$
\begin{aligned}
J_1 &\le C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\Big(\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\Big(\sum_{|k-j|\le 4}2^{k(2\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\Big)^{\frac{1}{q}}\\
&\le C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{|k-j|\le 4}2^{(j-k)(-1+\frac{n}{p'}+\frac{\lambda}{p})q}\,2^{k(2+\frac{n}{p'}+\frac{\lambda}{p}-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{g}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\Big)^{\frac{1}{q}}\\
&\le C\|f\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\|g\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})},
\end{aligned}
$$

where we have used $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$.
Similary, we get

$$J_2 \le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.$$

For J_3, first we use the Young inequality in Morrey spaces, the Bernstein inequality ($|\gamma|=0$) together with the Hölder inequality, to get

$$
\begin{aligned}
\|\widehat{I_j^3}\|_{L^1(I,M_p^\lambda)} &\le \sum_{k\ge j-3}\|\widehat{(\dot{\Delta}_k f\tilde{\dot{\Delta}}_k\nabla g)}\|_{L^1(I,M_p^\lambda)}\\
&= \sum_{k\ge j-3}\|\widehat{(\dot{\Delta}_k f}*\widehat{\tilde{\dot{\Delta}}_k\nabla g)}\|_{L^1(I,M_p^\lambda)}\\
&\le \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{|l-k|\le 1}\|\varphi_l\widehat{\nabla g}\|_{L^{\rho_0}(I,L^1)}\\
&\le C\sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\sum_{|l-k|\le 1}2^l2^{l(\frac{n}{p'}+\frac{\lambda}{p})}\|\varphi_l\widehat{g}\|_{L^{\rho_0}(I,M_p^\lambda)}\\
&\le C\sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big(\sum_{|l-k|\le 1}2^{l(\alpha-1-\frac{\alpha}{\rho_0})q'}\Big)^{\frac{1}{q'}}\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\\
&\le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0}})}\sum_{k\ge j-3}2^{k(\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}
\end{aligned}
$$

Then, applying the Hölder inequality for series, and noticing that $\lambda > n-(n+3-2\alpha)p$ implies that $3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}>0$, we obtain

$$
\begin{aligned}
J_3 &\le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\Big(\sum_{j\in\mathbb{Z}}2^{j(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\Big(\sum_{k\ge j-3}2^{k(\alpha-1-\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\Big)^{\frac{1}{q}}\\
&\le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\ge j-3}2^{(j-k)(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})}\,2^{k(2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})}\|\varphi_k\widehat{f}\|_{L^{\rho_0'}(I,M_p^\lambda)}\Big)^q\Big)^{\frac{1}{q}}\\
&\le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{-2+\frac{n}{p'}+\frac{\lambda}{p}+\frac{2}{\rho_0'}})}\sum_{i\le 3}2^{i(3-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})}\\
&\le C\|g\|_{\mathfrak{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}\|f\|_{\mathfrak{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.
\end{aligned}
$$

Thus, we finished the proof of Lemma 4.1.

V. Proof of Theorem 1.1

To ensure the existence of the global and local solution of the system (1), we will use Lemma 2.3 with the linear and bilinear estimate that we have established in section 3 and 4.

Let $\rho_0 > \dfrac{\alpha}{\alpha-1}$ be any given real number and $\frac{1}{\rho_0}+\frac{1}{\rho_0'}=1$. Note that the space X_T defined in Theorem 1.1 is a Banach space equipped with the norm

$$\|u\|_{X_T} = \|u\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})}+\|u\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})}.$$

We first prove global existence for small initial data. For this purpose we choose $T = \infty$.
Set

$$B_1(v,w) := -\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

$$B_2(v,w) := \int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}} \nabla\cdot\left(w\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau,$$

Then the equivalent integral system (1.2) can be rewritten as

$$(v(t),w(t)) = (e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)+(B_1(v,w),B_2(v,w)). \quad (13)$$

According to Lemma 3.1 with $s = 2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}$, $I=[0,\infty)$ and $\rho=\rho_0$ $(or\ \ \rho_0')$, we obtain

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \leq C_0\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

and

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \leq C_0\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}},$$

which implies

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0\|_{X_\infty} \leq 2C_0\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Similary,

$$\|e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0\|_{X_\infty} \leq 2C_1\|w_0\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Thus

$$\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq C_2\|(v_0,w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \quad (14)$$

Applying Lemma 3.2 with $s = 2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}$ and $\rho_1 = 1$, and Lemma 4.1, we obtain

$$\begin{aligned}
&\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \\
&= \|\int_0^t e^{-(t-\tau)(-\Delta)^{\frac{\alpha}{2}}}\nabla \cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)(\tau)d\tau\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \\
&\lesssim \|\nabla\cdot\left(v\nabla(-\Delta)^{-1}(w-v)\right)\|_{\mathfrak{L}^1\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \times\|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \\
&\quad +\|(-\Delta)^{-1}(w-v)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \times\|v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \\
&\lesssim \|v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \times\|w-v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \\
&\quad +\|w-v\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \times\|v\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \\
&\leq C_3\Big(\|(v,w)\|_{\mathfrak{L}^{\rho_0}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}}\right)} \times\|(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)}\Big) \\
&\leq C_3\|(v,w)\|^2_{X_\infty}
\end{aligned}$$

Analogously, we get

$$\|B_1(v,w)\|_{\mathfrak{L}^{\rho_0'}\left(0,\infty;\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}}\right)} \leq C_3\|(v,w)\|^2_{X_\infty}$$

Thus, we obtain

$$\|B_1(v,w)\|_{X_\infty} \leq 2C_3\|(v,w)\|^2_{X_\infty}$$

Similary,

$$\|B_2(v,w)\|_{X_\infty} \leq 2C_4\|(v,w)\|^2_{X_\infty}$$

Finally,

$$\|(B_1(v,w),B_2(v,w))\|_{X_\infty} \leq C\|(v,w)\|^2_{X_\infty}$$

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By Lemma 2.3, we know that if $\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq \varepsilon$ with $\varepsilon = \frac{1}{4C}$, then the system (1) has a unique global solution in $\bar{B}(0, 2\varepsilon) = \{x \in X_\infty \ \|x\|_{X_\infty} \leq 2\varepsilon\}$. To prove $\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq \varepsilon$, according to (14) we have

$$\|(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0)\|_{X_\infty} \leq C_2 \|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

So, if $\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq K$ with $K = \frac{1}{4CC_2}$, then (1) has a unique global solution $(v, w) \in X_\infty$ satisfying

$$\|(v, w)\|_{X_\infty} \leq \frac{1}{2C}$$

For the local existence, we shall decompose the initial data v_0 into two terms

$$v_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{v}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{v}_0\right) := v_{0,1} + v_{0,2},$$

where $\delta = \delta(v_0) > 0$ is a real number. Similary, we decompose w_0

$$w_0 = \mathcal{F}^{-1}\left(\chi_{B(0,\delta)}\hat{w}_0\right) + \mathcal{F}^{-1}\left(\chi_{B^C(0,\delta)}\hat{w}_0\right) := w_{0,1} + w_{0,2}.$$

Since

$$\begin{cases} v_{0,2} \longrightarrow 0 \ \text{in} \ \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{when} \quad \delta \to +\infty, \\ w_{0,2} \longrightarrow 0 \ \text{in} \ \mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}} \ \text{when} \quad \delta \to +\infty, \end{cases}$$

then, there exists δ large enough such that

$$C_2\|(v_{0,2}, w_{0,2})\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{2}.$$

We get

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_0\right)\right\|_{X_T} \leq \frac{\varepsilon}{2} + \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T}$$

We have

$$\begin{aligned} \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T} = &\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \\ &+ \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})} \end{aligned}$$

Using the fact that $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we have

$$\begin{aligned} &\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \\ &= \left\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})q}\|\varphi_j \widehat{e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\right\}^{1/q} \\ &+\left\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0})q}\|\varphi_j \widehat{e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}}\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\right\}^{1/q} \\ &= \left\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{\alpha}{\rho_0})q}\|\varphi_j|\xi|^\alpha\chi_{B(0,\delta)}\hat{v}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\right\}^{1/q} \\ &+\left\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}2^{j(\frac{\alpha}{\rho_0})q}\|\varphi_j|\xi|^\alpha\chi_{B(0,\delta)}\hat{w}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\right\}^{1/q} \\ &\lesssim \delta^{\alpha+\frac{\alpha}{\rho_0}}\left(\Big(\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\hat{v}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\Big)^{1/q}\right. \\ &\left.+\left\{\sum_{j\in\mathbb{Z}} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q}\|\varphi_j\hat{w}_0\|^q_{L^{\rho_0}(I,\mathrm{M}_p^\lambda)}\right\}^{1/q}\right) \\ &\leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \end{aligned}$$

Thus

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0}})} \leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}$$

Similary,

$$\left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{\mathcal{L}^{\rho_0'}(I,\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}+\frac{\alpha}{\rho_0'}})} \leq C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}}.$$

Hence,

$$\begin{aligned} \left\|\left(e^{-t(-\Delta)^{\frac{\alpha}{2}}}v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}}w_{0,1}\right)\right\|_{X_T} \leq & C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \\ &+C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \end{aligned}$$

Then, if we choose T small enough such that

$$\begin{cases} C_5\delta^{\alpha+\frac{\alpha}{\rho_0}}T^{\frac{1}{\rho_0}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{4} \\ \textit{and} \\ C_5\delta^{\alpha+\frac{\alpha}{\rho_0'}}T^{\frac{1}{\rho_0'}}\|(v_0, w_0)\|_{\mathcal{FN}_{p,\lambda,q}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}} \leq \frac{\varepsilon}{4}. \end{cases}$$

i.e,

$$\begin{cases} T \leq \left(\dfrac{\varepsilon}{4C_5 \delta^{\alpha + \frac{\alpha}{\rho_0}} \|(v_0, w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \right)^{\rho_0} \\ and \\ T \leq \left(\dfrac{\varepsilon}{4C_5 \delta^{\alpha + \frac{\alpha}{\rho_0'}} \|(v_0, w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \right)^{\rho_0'} \end{cases}$$

So, if we choose

$$T \leq \min \Big(\Big(\frac{\varepsilon}{4C_5 \delta^{\alpha + \frac{\alpha}{\rho_0}} \|(v_0, w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \Big)^{\rho_0}, \Big(\frac{\varepsilon}{4C_5 \delta^{\alpha + \frac{\alpha}{\rho_0'}} \|(v_0, w_0)\|_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}} \Big)^{\rho_0'} \Big)$$

then

$$\left\| \left(e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_{0,1}, e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_{0,1} \right) \right\|_{X_T} \leq \frac{\varepsilon}{2}.$$

This result with (5.2) yields that

$$\left\| \left(e^{-t(-\Delta)^{\frac{\alpha}{2}}} v_0, e^{-t(-\Delta)^{\frac{\alpha}{2}}} w_0 \right) \right\|_{X_T} \leq \varepsilon.$$

Thus for any arbitrary $(v_0, w_0) \in \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, (1) has a unique local solution in $\bar{B}(0, 2\varepsilon) = \{x \in X_T \ : \ \|x\|_{X_T} \leq 2\varepsilon\}$.

Regularity:

We know if $(v, w) \in X_T \times X_T$ is a solution of (1), then we can show that

$$\nabla \cdot (v \nabla \phi), \ \ \nabla \cdot (w \nabla \phi) \in \mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right).$$

By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned} & \|v(t_1) - v(t_2)\|^q_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ \leq & \sum_{j \leq N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^{\lambda}_p} \right)^q \\ & + 2 \sum_{j > N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t)\|_{L^\infty(I, \mathrm{M}^{\lambda}_p)} \right)^q, \end{aligned}$$

where $\hat{v}_j = \varphi_j \hat{v}$. For any small constant $\varepsilon > 0$, let N be large enough such that

$$\sum_{j > N} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q} \|\hat{v}_j(t)\|^q_{L^\infty(I, \mathrm{M}^{\lambda}_p)} \leq \frac{\varepsilon}{4}.$$

According to Taylor's formula and using the same arguments as [[21], Proposition 2.3], we get

$$\begin{aligned} & \sum_{j \leq N} \left(2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})} \|\hat{v}_j(t_1) - \hat{v}_j(t_2)\|_{\mathrm{M}^{\lambda}_p} \right)^q \\ \lesssim & |t_1 - t_2|^q \sum_{j \leq N} 2^{j(2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p})q} \left\| (\hat{\partial_t u})_j \right\|^q_{L^1(I, \mathrm{M}^{\lambda}_p)} \\ \lesssim & |t_1 - t_2|^q \times \|\partial_t u\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \\ \lesssim & |t_1 - t_2|^q \times \Big(\|\Delta v\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \\ & + \|\nabla \cdot (v \nabla \phi)\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \Big) \\ \lesssim & |t_1 - t_2|^q \times \Big(\|v\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \\ & + \|\nabla \cdot (v \nabla \phi)\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \Big) \\ \lesssim & |t_1 - t_2|^q \times \Big(\|v_0\|^q_{\mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ & + 2 \|\nabla \cdot (v \nabla \phi)\|^q_{\mathfrak{L}^1 \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)} \Big). \end{aligned}$$

Thus, we obtain the continuity of v in time t.
Similary, we use the same discusion to get the continuity of w in time t.
Hence $(v, w) \in C \left(0, T; \mathcal{FN}^{2-2\alpha+\frac{n}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \right)$, and we are done.

REFERENCES

[1] A., Naoufel Ben, F. Méhats, and N. Vauchelet, A note on the long time behavior for the drift-diffusion-Poisson system, Comptes Rendus Mathematique, Vol.339, pp.683-688, 2004.
[2] B., Hajer, The Littlewood-Paley theory: a common thread of many words in nonlinear analysis, 2019.
[3] H. Bahouri, J. Chemin, and R. Danchin. Fourier analysis and nonlinear partial differential equations, Springer Science and Business Media, Vol.343, 2011.
[4] Biler, Piotr, Existence and asymptotics of solutions for a parabolic-elliptic system with nonlinear no-flux boundary conditions, Nonlinear Analysis: Theory, Methods and Applications Vol.19, pp.1121-1136, 1992.
[5] B., Piotr, and J. Dolbeault, Long time behavior of solutions to Nernst-Planck and Debye-Hückel drift-diffusion systems, Annales Henri, Vol.1, No.3, 2000.
[6] B., Piotr, D. Hilhorst, and T. Nadzieja, Existence and nonexistence of solutions for a model of gravitational interaction of particles, II., Colloquium Mathematicum, Vol.67, Instytut Matematyczny Polskiej Akademii Nauk, 1994.
[7] Biler, Piotr, Waldemar Hebisch, and Tadeusz Nadzieja, The Debye system: existence and large time behavior of solutions, Nonlinear Analysis: Theory, Methods and Applications, Vol.23, pp.1189-1209, 1994.
[8] C., Xiaoli, Well-Posedness of the Keller–Segel System in Fourier–Besov–Morrey Spaces, Zeitschrift für Analysis und ihre Anwendungen, Vol.37, pp.417-434, 2018.
[9] D., Chao, and C. Li, Endpoint bilinear estimates and applications to the two-dimensional Poisson–Nernst–Planck system, Nonlinearity, Vol.26, 2013.
[10] F., Lucas CF, and L. SM Lima, Self-similar solutions for active scalar equations in Fourier–Besov–Morrey spaces, Monatshefte für Mathematik, Vol.175, pp.491-509, 2014.

[11] H., Gajewski, On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, Vol.65, pp.101-108, 1985.
[12] Iwabuchi, Tsukasa, Global well-posedness for Keller–Segel system in Besov type spaces, Journal of Mathematical Analysis and Applications, Vol.379, pp.930-948, 2011.
[13] I., Tsukasaa, and M. Nakamura, Small solutions for nonlinear heat equations, the Navier-Stokes equation and the Keller-Segel system in Besov and Triebel-Lizorkin spaces, Advances in Differential Equations, pp.687-736, 2013.
[14] Karch, G., Scaling in nonlinear parabolic equations, Journal of mathematical analysis and applications, Vol.234, pp.534-558, 1999.
[15] K., Tosio, Strong solutions of the Navier-Stokes equation in Morrey spaces, Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society, Vol.22, 127-155, 1992.
[16] K., Masaki, and T. Ogawa, Well-posedness for the drift-diffusion system in Lp arising from the semiconductor device simulation, Journal of Mathematical Analysis and Applications Vol.342, pp.1052-1067, 2008.
[17] L., Rieusset, P. Gilles, Recent developments in the Navier-Stokes problem. CRC Press, 2002.
[18] M. Cannone, G. Wu, Global well-posedness for Navier-Stokes equations in critical Fourier-Herz spaces, Nonlinear Anal., 75, 2012.
[19] O., Takayoshi, and S. Shimizu, The drift–diffusion system in two-dimensional critical Hardy space, Journal of Functional Analysis Vol.255, 1107-1138, 2008.
[20] S., Siegfried, Analysis and simulation of semiconductor devices. Springer Science and Business Media, 2012.
[21] Z., Jihong, Q. Liu, and S. Cui, Existence of solutions for the Debye-Hückel system with low regularity initial data, Acta applicandae mathematicae, Vol.125, 1-10, 2013.

Existance results for a neutral functionnal integrodifferential inclusion with finite delay

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***Abstract*— This paper is mainly concerned with existence of mild solution for a neutral functionnal integrodiferential inclusion with finit delay. The results are bobtained by using a fixed point theoreme for condensing multivalued maps.**

***Index Terms*—integrodifferentional inclusion, Selection, Fixed point theory, neutral functional differential and integrodifferential inclusion, convex multivalued map.**

I. INTRODUCTION:

In this paper we prove the existence of mild solution, for a neutral functional integrodifferentional inclusion with finite delay. In section 2 we will recall briefly some basic definitions and preliminary facts which will be used in the following section. Section 3 deals with the existence of mild solution for a neutral functional integrodifferentional inclusion with finite delay of the forme :

$$\begin{cases} \dfrac{d}{dt}\mathcal{F}(t,u_t) \in A\mathcal{F}(t,u_t) + \displaystyle\int_0^t B(t-s)\mathcal{F}(s,u_s)ds \\ +G(t,u_t) \quad \text{for} \quad t \in [0,b] \\ u_0 = \varphi(\theta) \quad \text{for} \quad \theta \in J_0 = [-r,0], \end{cases} \tag{1}$$

where $\big(A, \mathcal{D}(A)\big)$ is the infinitesimal generator of a compact resolvant operator $R(t)$, $t \geq 0$, in Banach space X, for $t \geq 0$ $B(t)$ is a closed linear operator with domain $\mathcal{D}(B)$, such that $\mathcal{D}(A) \subset \mathcal{D}(B)$. $G : J \times \mathcal{C}(J_0, X) \longrightarrow 2^X$ ($J_0 = [-r, 0]$), is a bounded, closed, convex, multivalued map and X a real Banach space.

For any continus function u defined on $J_1 = [-r, b]$, and any $t \in J$, we denot by u_t the element of $\mathcal{C}(J_0, X)$ defined by:

$$u_t(\theta) = u(t+\theta) = \varphi(\theta), \quad \theta \in J_0 = [-r, 0],$$

Here $u_t(.)$ represents the history of the state from time $t-r$, up to the present time t, and $\mathcal{F} : J \times \mathcal{C}(J_0, X) \longrightarrow X$ defined by :

$$\mathcal{F}(t,\varphi) = \varphi(0) - F(t,\varphi) = u(t) - F(t,u_t), \quad \forall (t,\varphi) \in J \times \mathcal{C}(J_0, X),$$

Where $F : J \times \mathcal{C}(J_0, X) \longrightarrow X$.
Wen $B = 0$ we refer to the paper of K.HILAL and K.EZZINBI [1] and the paper of K.EZZINBI and X.FU [2].
This paper is motivated by the recents results of [1] and BENCHOHRA [3]. Here we compose the above results and prove the existence of mild solution for our probleme (1), relying on a fixed point theorem for condensing maps due to Martelli [4].

II. PRELIMINARIES:

In this section, we introduce some basic definitions, notations, and lemmas that are used throughout this paper.

$\mathcal{C}(J, X)$ is the Banach space of continuous functions from J into X with the norm :

$$\|u\|_\infty := \sup\{|u(t)|; t \in J\}$$

A measurable function $u : J \longrightarrow X$ is Bochner integrable if and only if $|u|$ is Lebesgue integrable (For properties of the Bochner integral see Yosida [5]).
$L^1(J, X)$ denotes the Banach space of continuous functions $u : J \longrightarrow X$ which are Bochner integrable normed by :

$$\|u\|_{L^1} := \int_0^T |u(t)|dt \quad \text{for all } u \in L^1(J, X)$$

Lemma 2.1: :
Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \longrightarrow 2^X$ is convex closed, if $G(x)$ is convex closed, for all $x \in X$; and G is bounded on bounded sets, if $G(B) = \underset{x\in B}{\cup} G(x)$ is bounded in X, for any bounded set B of X.

Theorem 2.1: :
G is said to be completely continuous if $G(B)$ is relatively compact, for every bounded subset $B \subset X$.

Theorem 2.2: :
G is called upper semi-continuous (u.s.c) on X, if for each $x \in X$, the set $G(x)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x)$, there exists an open neighborhood V of x such that $G(V) \in B$.

Lemma 2.2: :
If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph $\Big($i.e $x_n \longrightarrow x, y_n \longrightarrow y; y_n \in G(x_n)$ imply $y \in G(x)\Big)$.

Definition 2.1: :
an upper semi-continuous multivalued map $G : X \longrightarrow X$ is said to be condensing if for any subset $B \subset X$ with $\alpha(B) \neq 0$, we have $\alpha\big(G(B)\big) < \alpha\big(B\big)$, where α denotes the Kuratowski measure of noncompactness [6].

Lemma 2.3: :

A completely continuous multivalued map is a condensing map

Theorem 2.3: :(Arzela–Ascoli's theorem)

Let K be a compact space and (E,d) a metric space. $A \subset \mathcal{C}(K,E)$ is relatively compact (i.e. included in a compact) if and only if, for any x of K:

- A is equicontinuous in x, i.e. for everything $\varepsilon > 0$, there exist a neighborhood V of x such that : $\forall f \in A, \forall y \in V \quad d(f(x), f(y)) < \varepsilon$
- The set $A(x) = \{f(x); f \in A\}$ is relatively compact.

In the following $\mathcal{BCC}(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X

Theorem 2.4: :(Leray-schauder's fixed point)

Let X be a Banach space and $N : X \longrightarrow \mathcal{BCC}(X)$ an u.s.c condensing map. If the set :

$$\Omega := \{u \in X : \lambda u \in Nu \quad \text{for} \quad \lambda > 1\}$$

is bounded, then N has a fixed point.

Definition 2.2: :(Finite delay differential equation)

Let $r > 0$; and $\mathcal{C}_r = \mathcal{C}\big([-r,0],\mathbb{R}^n\big)$, the Banach space of continuous functions,
$\varphi : [-r,0] \longrightarrow \mathbb{R}^n$ with $\|\varphi\|_\infty = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\|$. We denot by u_t element of $\mathcal{C}_r$ defined by :

$$u_t(\theta) = u(t+\theta) = \varphi(\theta), \quad \theta \in J_0 = [-r,0],$$

Let $f : \mathbb{R}^+ \times \mathcal{C}_r \longrightarrow \mathbb{R}$, a general form of the finit-delay differential equation is :

$$\frac{d}{dt}u(t) = f(t,u_t)$$

Definition 2.3: :(Resolvent operator [2])

A family of bounded linear operators $R(t) \in B(X)$, $\big(B(X)$ is the Banach space of all linear bounded operator from X into $X\big)$, for $t \in J$ is called a resolvent operator for :

$$\frac{du}{dt} = Au(t) + \int_0^t f(t-s)u(t)ds$$

If:

1- $R(0) = I$, the identity operator on X, and $\|R(t)\| \leqslant M$ with $M > 1$.
2- For all $u \in X$; $R(t)u$ is continuous for $t \in J$
3- $R(t) \in B(Y)$; $t \in J$; where Y is the Banach space formed from $\mathcal{D}(A)$, for $y \in Y, R(.)y \in \mathcal{C}^1(J,X) \cap \mathcal{C}(J,Y)$ and :

$$R^{'}(t)y = AR(t)y + \int_0^t f(t-s)R(s)yds = R(t)Ay + \int_0^t R(t-s)f(s)yds.$$

III. EXISTANCE RESULTS :

In order to define the concept of mild solution for (1), by comparaison with the evolution problem

$$\frac{dv}{dt} = Av(t) + \int_0^t f(t-s)v(t)ds + h(t) \quad ; \quad v(0) = a$$

We asssociate (1) to the integral equation :

$$u(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t-s)g(s)\mathrm{d}s \quad t \in [0,b]$$

Where $g \in \mathcal{S}_{G,u} = \{g \in L^1(J,X) : g(t) \in G(t,u_t); \quad t \in J\}$

Definition 3.1: :

A function $u \in \mathcal{C}([-r,b],X)$ is called a mild solution of (1) if :

1- $u(0) = \varphi(\theta); \quad \theta \in [-r,0]$.
2- There exist a function $g \in \mathcal{S}_{G,u}$ such that :

$$u(t) = R(t)\mathcal{F}(0,\ \varphi) + F(t,\ u_t) + \int_0^t R(t-s)g(s)\mathrm{d}s \quad t \in [0,b]$$

Where, $\mathcal{F}(0,\ \varphi) = \varphi(\theta) - F(t,\varphi)$

Assume that :

(H1)- A is the infinitesimal generator of a compact resolvent operator $R(t)$ in X such that :

$$\|R(t)\| \leq M_1 \quad for \quad some \quad M_1 \geq 1 \quad ; \quad t \in J$$

(H2)- There exists constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that :

$$|F(t,u)| \leq c_1\|u\| + c_2; \quad t \in J \quad u \in \mathcal{C}(J_0,X)$$

(H3)- $\varphi \in \mathcal{C}([-r,0],X)$ is completely continuous and there exists a constant M_2 such that:

$$\|\varphi\| \leq M_2$$

(H4)- $G : J \times \mathcal{C}(J_0,X) \longrightarrow \mathcal{BCC}(X)$; $(t,u) \longrightarrow G(t,u)$ is measurable with respect to t for each $u \in \mathcal{C}(J_0,X)$, u.s.c with respect to u for each $t \in J$; and for each fixed $u \in \mathcal{C}(J_0,X)$ the set :

$$\mathcal{S}_{G,u} = \{g \in L^1(J,X) : g(t) \in G(t,u_t); \quad t \in J\}$$

is nonempty.

(H5)- $\|G(t,u)\| := sup\{|g| : g \in G(t,u)\} \leq p(t)\Psi(\|u\|)$ for all $t \in J$ and all $u \in \mathcal{C}(J_0,X)$, where $p \in L^1(J,\mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow [0,+\infty)$ is continuous and increasing with :

$$\int_0^b \omega(s)ds < \int_c^\infty \frac{d\tau}{\tau + \Psi(\tau)}$$

Where

$$c = \frac{1}{1-c_1}\{M_1\big(M_2(1+c_1)+c_2\big)+c_2\} \quad and \quad \omega(s) = \frac{1}{1-c_1}M_1p(t)$$

(H6)- The function F is complety continuous and for any bounded set $B \subseteq \mathcal{C}(J_1,X)$ the set $\{t \longrightarrow F(t,u_t) : u \in B\}$ is equicontinuous in $\mathcal{C}$.

The following lemma is crucial in the proof of our existence results.

Lemma 3.1: :

Let I be a compact real interval and X be a Banach space. Let G be a multivalued map satisfying $(H4)$. And let Γ be a linear continuous mapping from $L^1(I,X)$ to $\mathcal{C}(J,X)$. Then the operator :

$$\Gamma \circ \mathcal{S}_G : \mathcal{C}(I,X) \longrightarrow \mathcal{BCC}\big(\mathcal{C}(I,X)\big); \quad u \longrightarrow \big(\Gamma \circ \mathcal{S}_G\big)(u) = \Gamma\big(\mathcal{S}_G\big)$$

Is closed graph operator in $\mathcal{C}(I,X) \times \mathcal{C}(I,X)$

Our main result may be presented as the following theorem.

Theorem 3.1: :

Assume that hypotheses $(H1) - (H6)$ hold, then the problem (1) has at least one mild solution on J_1.

Proof 3.1: :

Let $\mathbf{C} := \mathcal{C}(J_1, X)$ be the Banach space of continuous function from J_1 into X endowed with the sup-norm :

$$\|u\|_\infty := sup\{|u| : t \in [-r, b]\}; for \quad u \in \mathbf{C}$$

Transform the problem into a fixed point problem. Consider the multivalued map, $\mathcal{N} : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ defined by :

$$\mathcal{N}u := \Bigg\{ h \in \mathbf{C} :$$

$$h(t) = \begin{cases} \varphi(t); t \in J_0 \\ R(t)\mathcal{F}(0, \ \varphi) + F(t, \ u_t) \\ + \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J \end{cases} \Bigg\}$$

Where : $g \in \mathcal{S}_{G,u} = \{g \in L^1(J, X) : g(t) \in G(t, u_t); \quad t \in J\}$

We have that the fixed points of N are mild solutions to (1). Now we shall prove that $\mathcal{N}$ is a completely continuous multivalued map, u.s.c, with convex closed values. The proof will be given in several steps.

Step 1 : $\mathcal{N}u$ is convex for each $u \in \mathbf{C}$.
Indeed, if h_1, h_2 belong to $\mathcal{N}u$, then there exist $g_1, g_2 \in \mathcal{S}_{G,u}$ such that for each $t \in J$ we have:

$$h_1(t) = R(t)\mathcal{F}(0, \ \varphi) + F(t, \ u_t) + \int_0^t R(t-s)g_1(s)\mathrm{d}s$$

and

$$h_2(t) = R(t)\mathcal{F}(0, \ \varphi) + F(t, \ u_t) + \int_0^t R(t-s)g_2(s)\mathrm{d}s$$

Let $0 \leq k \leq 1$. Then for each $t \in J$ we have :

$$\Big(kh_1 + (1-k)h_2\Big)(t) = R(t)\mathcal{F}(0, \ \varphi) + F(t, \ u_t) + \int_0^t R(t-s)\Big(kg_1(s) + (1-k)g_2(s)\Big)\mathrm{d}s$$

Thus $kh_1 + (1-k)h_2 \in \mathcal{N}u$ $\big($ because $\mathcal{S}_{G,u}$ is convex$\big)$, then $\mathcal{N}u$ is convex for each $u \in \mathbf{C}$

Step 2 : We will prove that $\mathcal{N}$ is a completely continuous operator. Using $(H6)$ it suffices to show that the operator $\mathcal{N}_1 : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ defined by : $\mathcal{N}_1 u := \Bigg\{ h_1 \in \mathbf{C} :$

$$h_1(t) = \begin{cases} \varphi(t); t \in J_0 \\ R(t)\mathcal{F}(0, \ \varphi) \\ + \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J \end{cases} \Bigg\}$$

is completly continuous .

i- $\mathcal{N}_1$ map bounded set into bounded set in $\mathbf{C}$:
Indeed, it is enough to show that there exists a positive constant l such that for each $h_1 \in \mathcal{N}_1 u$; $u \in \mathbf{B}_q = \{u \in \mathbf{C} : \|u\|_\infty \leq q\}$ we have $\|h_1\|_\infty \leq l$.
If $h_1 \in \mathcal{N}_1 u$ then there exist $g \in \mathcal{S}_{G,u}$, such that for every $t \in J$ we have :

$$h_1(t) = R(t)\mathcal{F}(0, \ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s$$

By $(H1) - (H3)$, and $(H5)$ we have for each $t \in J$:

$$\begin{aligned} |h_1(t)| &\leq \|R(t)\mathcal{F}(0, \ \varphi)\| + \int_0^t \|R(t-s)g(s)\|\mathrm{d}s \\ &\leq M_1[c_1 M_2 + c_2] \\ &+ M_1 \sup_{u \in [0,q]} \Psi(u)\Big(\int_0^t p(s)\mathrm{d}s\Big) \end{aligned}$$

Then for each $h \in \mathcal{N}_1(B_q)$:

$$\begin{aligned} \|h_1(t)\|_\infty &\leq M_1[c_1 M_2 + c_2] \\ &+ M_1 \sup_{u \in [0,q]} \Psi(u)\Big(\int_0^b p(s)\mathrm{d}s\Big) \end{aligned}$$

Then $\mathcal{N}_1$ is bounded.

ii- $\mathcal{N}_1$ maps bounded set into equicontinuous sets of $\mathbf{C}$:

Let $\tau_1, \tau_2 \in J; \tau_1 < \tau_2$, and B_q be bounded set of $\mathbf{C}$; for each $u \in B_q$ and $h_1 \in \mathcal{N}_1 u$; there exist $g \in \mathcal{S}_{G,u}$ such that :

$$h_1(t) = R(t)\mathcal{F}(0, \ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

Thus,

$$\begin{aligned} h_1(\tau_2) - h_1(\tau_1) &= R(\tau_2)\mathcal{F}(0, \ \varphi) \\ &+ \int_0^{\tau_2} R(\tau_2 - s)g(s)\mathrm{d}s - R(\tau_1)\mathcal{F}(0, \ \varphi) \\ &- \int_0^{\tau_1} R(\tau_1 - s)g(s)\mathrm{d}s \\ &= \Big(R(\tau_2) - R(\tau_1)\Big)\mathcal{F}(0, \ \varphi) \\ &+ \int_0^{\tau_1} \Big(R(\tau_2 - s) - R(\tau_1 - s)\Big)g(s)\mathrm{d}s \\ &+ \int_{\tau_1}^{\tau_2} R(\tau_2 - s)g(s)\mathrm{d}s \end{aligned}$$

Then

$$\begin{aligned} \|h_1(\tau_2) - h_1(\tau_1)\| &\leq \|R(\tau_2) - R(\tau_1)\| \\ &+ \int_0^{\tau_1} \|R(\tau_2 - s) \\ &- R(\tau_1 - s)\|\|g(s)\|ds \\ &+ \int_{\tau_1}^{\tau_2} \|R(\tau_2 - s)\|\|g(s)\|ds \end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{N}_1 u$ is equicontinuous on J_1

iii- $V(t) = \{h_1(t); h_1 \in \mathcal{N}_1(B_q)\}$ is relatively compact on X:
By $(H4)$ $V(t)$ is relatively compact for $t = 0$; let $0 \leq t \leq b$ be fixed and let ε be a real number satisfying $0 \leq \varepsilon < t$ for $u \in B_q$ and $g \in \mathcal{S}_{G,u}$ such that :

$$h_1(t) = R(t)\mathcal{F}(0, \ \varphi) + \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

and,

$$h_{1,\varepsilon}(t) = R(t)\mathcal{F}(0,\ \varphi) + \int_0^{t-\varepsilon} R(t-s)g(s)\mathrm{d}s; \quad t \in J$$

The set $V_\varepsilon(t) = \{h_{1,\varepsilon}(t); h_{1,\varepsilon} \in \mathcal{N}_1(B_q)\}$ is relatively compact beceuse $R(t)$ is compact then;

$$\begin{aligned}\|h_1(t) - h_{1,\varepsilon}(t)\| &= \int_{t-\varepsilon}^{t} R(t-s)g(s)\| \\ &\leq M_1 \sup_{u\in[0,q]} \Psi(u) \int_0^t p(s)\mathrm{d}s.\varepsilon \\ &\leq r\varepsilon\end{aligned}$$

With; $r = M_1 \sup_{u\in[0,q]} \Psi(u) \int_0^b p(s)ds$, this implies that $V(t)$ is relativement compact thus by $(i),(ii),(iii)$ and by Arzela-Ascoli theorem, we can deduce that $\mathcal{N}_1$ is completely continuous then $\mathcal{N} : \mathbf{C} \longrightarrow 2^{\mathbf{C}}$ is completely continuous.

Step 3 : $\mathcal{N}$ has a closed graph.
Let $u_n \longrightarrow u$, $h_n \in \mathcal{N}u_n$ and $h_n \longrightarrow h$, we shall prove that $h \in \mathcal{N}u$.
$h_n \in \mathcal{N}u_n$ then there exists $g_n \in \mathcal{S}_{G,u_n}$ such that

$$\begin{aligned}h_n(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_{nt}) \\ &+ \int_0^t R(t-s)g_n(s)\mathrm{d}s; \quad t \in J\end{aligned}$$

We should prove that $g \in \mathcal{S}_{G,u}$ such that for each $t \in J$:

$$\begin{aligned}h(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_t) \\ &+ \int_0^t R(t-s)g(s)\mathrm{d}s; \quad t \in J\end{aligned}$$

Since F is continuous, we have that:

$$\begin{aligned}&\|\Big(h_n(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_{nt})\Big) \\ &- \Big(h(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_t)\Big)\|_\infty \longrightarrow 0\end{aligned}$$

As $n \longrightarrow \infty$.
Consider the linear operator :

$$\Gamma : \mathbf{L}^1(J, X) \longrightarrow \mathcal{C}(J, X)$$

$$g \longrightarrow \Gamma(g)(t) = \int_0^t R(t-s)g(s)ds$$

From Lemma 3.1; $\Gamma \circ \mathcal{S}_G$ is a closed graph operator then we have that :

$$h_n(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_{nt}) \in \Gamma\Big(\mathcal{S}_{G,u_n}\Big)$$

Since $u_n \longrightarrow u$, and by the lemma 3.1 :

$$h(t) - R(t)\mathcal{F}(0,\ \varphi) - F(t, u_t) \in \Gamma\Big(\mathcal{S}_{G,u}\Big)$$

It follows that $g \in \mathcal{S}_{G,u}$ such that

$$\begin{aligned}h(t) &= R(t)\mathcal{F}(0,\ \varphi) + F(t, u_t) \\ &+ \int_0^t R(t-s)g(s)\mathrm{d}s; t \in J\end{aligned}$$

From the Step 1, step 2 and step 3 we deduce that $\mathcal{N}$ is u.s.c, completely continuous then by lemma (2.3), $\mathcal{N}$ is a condensing, bounded, closed and convex operator. In order to prove that $\mathcal{N}$ has a fixed point, we need one more step.
Step 4 : The set $\Omega := \{u \in X : \lambda u \in \mathcal{N}u \quad \text{for} \quad \lambda > 1\}$ is bounded. Let $u \in \Omega$. Then $\lambda u \in \mathcal{N}u$, thus there exists $g \in \mathcal{S}_{G,u}$ such that :

$$\begin{aligned}u(t) &= \lambda^{-1}R(t)\mathcal{F}(0,\ \varphi) + \lambda^{-1}F(t, u_t) \\ &+ \lambda^{-1}\int_0^t R(t-s)g(s)\mathrm{d}s\end{aligned}$$

By $(H_1) - (H_3)$and (H_5) we have :

$$\begin{aligned}|u(t)| &\leq M_1\Big((1+c_1)M_2 + c_2\Big) + c_1\|u_t\| + c_2 \\ &+ M_1\int_0^t p(s)\Psi(\|u_s\|)ds\end{aligned}$$

Consider the function defined by :

$$\mu(t) = sup\{|u(s)| : -r \leq s \leq t\}; 0 \leq t \leq b$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |u(t^*)|$.
$\star$ If $t^* \in J_0 = [-r, 0]$ then :

$$\mu(t) \leq \|\phi\| \leq M_2$$

$\star\star$ If $t^* \in J = [0, b]$ then :

$$\begin{aligned}\mu(t) &\leq M_1\Big((1+c_1)M_2 + c_2\Big) + c_1\mu(t) + c_2 \\ &+ M_1\int_0^t p(s)\Psi\big(\mu(s)\big)ds\end{aligned}$$

then;

$$\begin{aligned}\mu(t) &\leq \frac{1}{1-c_1}\Bigg(M_1\Big((1+c_1)M_2 + c_2\Big) + c_2 \\ &+ M_1\int_0^t p(s)\Psi\big(\mu(s)\big)ds\Bigg)\end{aligned}$$

since $M_1 \geq 1$ Let us take the right-hand side of the above inequality as $\nu(t)$. Then we have.
$c = \nu(0) = \dfrac{1}{1-c_1}\Bigg(M_1\big(M_2(1+c_1) + c_2\big) + c_2\Bigg)$ and $\mu(t) \leq \nu(t) \quad ; \quad \forall t \in J$ then,

$$\nu'(t) = \frac{1}{1-c_1}M_1 p(t)\Psi\big(\mu(t)\big)$$

By using $H(5)$ we get :

$$\nu' < \omega(t)\Psi\big(\nu(t)\big)$$

This implies that

$$\int_{\nu(0)=c}^{\nu(t)} \frac{d\tau}{\Psi(\tau)} < \int_{0}^{b} \omega(s)ds < \int_{c}^{\infty} \frac{d\tau}{\tau + \Psi(\tau)}$$

This implies that there exists a constant K such that $\nu \leq K$, $t \in J$ and $\mu \leq K$, $t \in J$. Since for every $t \in J$ we have $\|u_t\| \leq \mu(t)$ then

$$\|u\|_\infty := sup\{|u(t)|; -r \leq t \leq b\} \leq K$$

Where K depends only on b and on the functions p and Ψ. This shows that Ω is bounded.
As a consequence of theorem 2.4 (Leray-schauders's fixed point) we deduce that $\mathcal{N}$ has a fixed point which is a solution of (1).

REFERENCES

[1] K. HILLAL and K. EZZINBI, Nonlocal differential inclusion with nondense domain and multivalued perturbations in Banach spaces.
[2] K. EZZINBI and X. FU, Existence and regularity of solutions for some neutral partial differential equations with nonlocal conditions.
[3] M. BENCHOHRA and S. K., NTOUYAS Nonlocal Cauchy Problems for Neutral Functional Differential and Integrodifferential Inclusions in Banach Spaces.
[4] M. Martelli, A Rothe's type theorem for non-compact acyclic-valued map, Boll. Un. Mat. Ital.
[5] K. Yosida, Functional Analysis. 6th ed. Springer, Berlin, 1980.
[6] J. Banas and K. Goebel, Measures of Noncompactness in Banach Spaces, Dekker, New York, 1980.

Fuzzy fractional differential wave equation

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Abstract—**This work is related to investigate integral solution of wave equation with fuzzy initial data under generalized fuzzy Caputo derivative. For the concerned investigation, we use the Fourier transform. The exact solution is given in the case of $\gamma = 2$. Some examples are presented to illustrate the results.**

Index Terms—**Generalized fuzzy derivative, Caputo fractional derivative, Hukuhara difference, fuzzy fourier transform.**

I. Introduction

The present paper investigate the analytic solution of the following problem

$$\begin{cases} {}_{gH}D_t^{\gamma} u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2} u(t,x) = 0, \\ -\infty < x < \infty, \ t \geq 0, \ 1 < \gamma < 2 \\ u(0,x) = a(x) \\ \frac{\partial}{\partial t} u(0,x) = b(x) \end{cases}$$

where a and b are two absolutely valued-functions in E^1. $-g$ is the generalized Hukuhara difference. ${}_{gH}D$ is the generalized fuzzy fractional caputo's derivative.

In 1965 L.Zadeh [13] introduced the basic ideas of the fuzzy set theory, as an extension of the classical notion of set. The authors in [6] give a generalization of the Hukuhara difference which guaranteed the existence of this is for two segments in $\mathbb{R}$. As consequance in the same work Bede and Stefanini presented the generalized derivative of a set valued-functions. Agarwal et al. [1] are the pioneers working in fuzzy fractional (DEs). They formulated the Riemann-Liouville differentiability notion as the base to define the concept of fuzzy fractional DEs. After that, they proved the existence of solutions of fuzzy fractional integral equations (IEs) under compactness type conditions using the Hausdorff measure of noncompactness in the paper [2]. Allahviranloo et all in [3] presented two new results on the existence of two kinds of $gH-$weak solutions of these problems and indicated the boundedness and continuous dependence of solutions on the initial data of the problems. In [5] the authors prove the existence and uniqueness theorems for non-linear fuzzy fractional Fredholm integro-differential equations under fractional generalized Hukuhara derivatives in the Caputo sense. From the idea of [5] we will try to prove the existence and uniqueness of fuzzy fractional wave equation.

This paper is organized as follows. In section 2 we recall some concepts concerning the fuzzy metric space. the generalized derivative take place in the section 3. In section 4 we give the concept of fuzzy Fourier transform and we presented some properties. We presented the solution of the fuzzy wave equation in section 5. Finally in section 6 two examples are given to illustrate the usefulness of our main results.

II. preliminaries

In this section, we present some definitions and introduce the necessary notation, which will be used throughout the paper.

We denote E^1 the class of function defined as follows:

$$E^1 = \left\{u : \mathbb{R} \to [0,1], \quad u \quad \text{satisfies } (1-4) \text{ below}\right\}$$

1) u is normal, i.e. there is a $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
2) u is a fuzzy convex set;
3) u is upper semi-continuous;
4) u closure of $\{x \in \mathbb{R}^n, \quad u(x) > 0\}$ is compact

For all $\alpha \in (0,1]$ the α-cut of an element of E^1 is defined by

$$u^\alpha = \left\{x \in \mathbb{R},\ u(x) \geq \alpha\right\}$$

By the previous properties we can write

$$u^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$$

By the extension principal of Zadeh we have

$$\begin{aligned} (u+v)^\alpha &= u^\alpha + v^\alpha; \\ (\lambda u)^\alpha &= \lambda u^\alpha \end{aligned}$$

For all $u, v \in E^1$ and $\lambda \in \mathbb{R}$

The distance between two element of E^1 is given by (see [4])

$$d(u,v) = \sup_{\alpha\in(0,1]} \max\left\{|\overline{u}(\alpha) - \overline{v}(\alpha)|, |\underline{u}(\alpha) - \underline{v}(\alpha)|\right\}$$

The metric space (E^1, d) is complete, separable and locally compact and the following properties for metric d are valid:

1) $d(u+v, u+w) = d(u,v)$;
2) $d(\lambda u, \lambda v) = |\lambda| d(u,v)$;
3) $d(u+v, w+z) \leq d(u,w) + d(v,z)$;

Remark II.1 *The space (E^1, d) is a linear normed space with $\|u\| = d(u,0)$.*

Definition II.2 *[10] A complex fuzzy number is a mapping $z : \mathbb{C} \to [0,1]$ with the following properties:*

1) *z is continuous;*
2) *z^α, $\alpha \in (0,1]$ is open, bounded, connected and simply connected;*
3) *z^1 is non-empty, compact, arcwise connected and simply connected.*

We denote the set of all fuzzy complex number by $\mathbb{C}^1$.

Definition II.3 *[6] The generalized Hukuhara difference of two fuzzy numbers $u, v \in E^1$ is defined as follows*

$$u -_g v = w \Leftrightarrow \begin{cases} u = v + w \\ or \quad v = u + (-1)w \end{cases}$$

In terms of α-levels we have

$$\left(u -_g v\right)^\alpha = \Big[\min\left\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\right\}, \max\left\{\underline{u}(\alpha) - \underline{v}(\alpha), \overline{u}(\alpha) - \overline{v}(\alpha)\right\}\Big]$$

and the conditions for the existence of $w = u -_g v \in E^1$ are

$$\text{case (i)} \quad \begin{cases} \underline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \\ \text{with} \quad \underline{w}(\alpha) \text{ increasing,} \\ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

$$\text{case (ii)} \quad \begin{cases} \underline{w}(\alpha) = \overline{u}(\alpha) - \overline{v}(\alpha) \text{ and } \overline{w}(\alpha) = \underline{u}(\alpha) - \underline{v}(\alpha) \\ \text{with} \quad \underline{w}(\alpha) \text{ increasing,} \\ \overline{w}(\alpha) \text{ decreasing}, \underline{w}(\alpha) \leq \overline{w}(\alpha) \end{cases}$$

for all $\alpha \in [0,1]$.

Throughout the rest of this paper, we assume that $u -_g v \in E^1$

Proposition II.4 *[11]*

$$\|u -_g v\| = d(u,v)$$

Since $\|.\|$ is a norm on E^n and by the proposition (II.4) we have

Proposition II.5

$$\|\lambda u -_g \mu u, 0\| = |\lambda - \mu| \|u\|$$

Let $f : [a,b] \subset \mathbb{R} \to E^1$ a fuzzy-valued function. The α-level of f is given by

$$f(x,\alpha) = \left[\underline{f}(x,\alpha), \overline{f}(x,\alpha)\right], \ \forall x \in [a,b], \ \forall \alpha \in [0,1].$$

Definition II.6 *[6] Let $x_0 \in (a,b)$ and h be such that $x_0 + h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy value function $f : (a,b) \to E^1$ at x_0 is defined as*

$$\lim_{h\to 0} \left\| \frac{f(x_0+h) -_g f(x_0)}{h} -_g f'_{gH}(x_0) \right\| = 0 \qquad \text{(II.1)}$$

If $f_{gH}(x_0) \in E^1$ satisfying (II.1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.

Definition II.7 *[6] Let $f : [a,b] \to E^1$ and $x_0 \in (a,b)$, with $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0. We say that*

1) *f is $[(i) - gH]$-differentiable at x_0 if*
$$f'_{i,gH}(x_0) = \left[\underline{f}'(x,\alpha), \overline{f}'(x,\alpha)\right] \qquad \text{(II.2)}$$
2) *f is $[(ii) - gH]$-differentiable at x_0 if*
$$f'_{ii,gH}(x_0) = \left[\overline{f}'(x,\alpha), \underline{f}'(x,\alpha)\right] \qquad \text{(II.3)}$$

Theorem II.8 *Let $f : J \subset \mathbb{R} \to E^1$ and $g : J \to \mathbb{R}$ and $x \in J$. Suppose that $g(x)$ is differentiable function at x and the fuzzy-valued function $f(x)$ is gH-differentiable at x. So*

$$(fg)'_{gH} = (f'g)_{gH} + (fg')_{gH}$$

Proof *Using (II.5), for h enough small we get*

$$\begin{aligned}
&\left\| \frac{f(x+h)g(x+h) -_g f(x)g(x)}{h} \right.\\
&\qquad \left. -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH} \right) \right\| \\
&= \left\| \frac{f(x+h)g(x+h) -_g f(x)g(x+h) + f(x)g(x+h) -_g f(x)g(x)}{h} \right.\\
&\qquad \left. -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH} \right) \right\| \\
&= \left\| \frac{(f(x+h) -_g f(x))g(x+h) + f(x)(g(x+h) -_g g(x))}{h} \right.\\
&\qquad \left. -_g \left((f'(x)g(x))_{gH} + (f(x)g'(x))_{gH} \right) \right\| \\
&\leq \left\| \frac{(f(x+h) -_g f(x))g(x+h)}{h} -_g \left((f'(x)g(x))_{gH} \right) \right\| \\
&\quad + \left\| \frac{(f(x)(g(x+h) -_g g(x))}{h} -_g \left((f(x)g'(x))_{gH} \right) \right\| \\
&\leq \left\| \frac{(f(x+h) -_g f(x))}{h} g(x+h) -_g \left((f'(x)g(x))_{gH} \right) \right\| \\
&\quad + \left\| f(x) \frac{((g(x+h) -_g g(x))}{h} -_g \left((f(x)g'(x))_{gH} \right) \right\|
\end{aligned}$$

which complet the proof by passing to limit.

Definition II.10 *[6] We say that a point $x_0 \in (a,b)$, is a switching point for the differentiability of f, if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that*

1) *type (1). at x_1 (II.2) holds while (II.3) does not hold and at x_2 (II.3) holds and (II.2) does not hold, or*
2) *type (2). at x_1 (II.3) holds while (II.2) does not hold and at x_2 (II.2) holds and (II.3) does not hold.*

Definition II.11 *[3] Let $f : (a,b) \to E^1$. We say that $f(x)$ is gH-differentiable of the 2nd-order at x_0 whenever the function $f(x)$ is gH-differentiable of the order $i, i = 0, 1$, at x_0, $((f(x_0))^{(i)}_{gH} \in E^1)$, moreover there isn't any switching point on (a,b). Then there exists $(f)''_{gH}(x_0) \in E^1$ such that*

$$\lim_{h\to 0} \left\| \frac{f'(x_0+h) -_g f'(x_0)}{h}, f''_{gH}(x_0) \right\| = 0$$

Definition II.12 *[3] Let $f : [a,b] \to E^1$ and $f'_g H(x)$ be gH-differentiable at $x_0 \in (a,b)$, moreover there isn't any switching point on (a,b) and $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ both differentiable at x_0. We say that*

- *f' is $[(i) - gH]$-differentiable at x_0 if*
$$f''_{i,gH}(x_0) = \left[\underline{f}''(x,\alpha), \overline{f}''(x,\alpha)\right]$$
- *f' is $[(ii) - gH]$-differentiable at x_0 if*
$$f''_{ii,gH}(x_0) = \left[\overline{f}''(x,\alpha), \underline{f}''(x,\alpha)\right]$$

Definition II.13 *[8] Let $f : [a,b] \to E^1$. We say that $f(x)$ is fuzzy Riemann integrable to $I \in E^1$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ with the norms $\Delta(P) < \delta$, we have*

$$d\left(\sum_p^{*} (v-u) f(\xi), I \right) < \epsilon$$

where $\sum_p^*$ denotes the fuzzy summation. We choose to write $I = \int_a^b f(x)dx$.

Theorem II.14 *[6] If f is gH-differentiable with no switching point in the interval $[a,b]$ then we have*

$$\int_a^b f(t)dt = f(b) -_g f(a)$$

Theorem II.15 *[12] Let $f(x)$ be a fuzzy-valued function on $(-\infty, \infty)$ and it is represented by $f(x,\alpha) = \left[\underline{f}(x,\alpha), \overline{f}(x,\alpha)\right]$ for any fixed $\alpha \in [0,1]$. Assume that $|\underline{f}(x,\alpha)|$ and $|\overline{f}(x,\alpha)|$ are Riemann integrable on*

$(-\infty,\infty)$ *for all* $\alpha \in [0,1]$. *Then* $f(x)$ *is improper fuzzy Riemann-integrable on* $(-\infty,\infty)$ *and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have*

$$\int_{-\infty}^{\infty} f(x)dx = \left[\int_{-\infty}^{\infty} \underline{f}(x,\alpha)dx, \int_{-\infty}^{\infty} \overline{f}(x,\alpha)dx\right]$$

From this theorem we can discuss the Fuzzy Riemann's improper integral

Lemma II.16 *Let* $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$, *given by* $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$, *and let* $a \in \mathbb{R}^+$
If $\int_a^{\infty} \underline{f}(x,t;\alpha)dt$ *and* $\int_a^{\infty} \overline{f}(x,t;\alpha)dt$ *are converges then*

$$\int_a^{\infty} f(x,t;\alpha)dt \in E^1$$

Proof *Just use the conditions* (II.1).

Theorem II.18 *Let* $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$ *be fuzzy-valued function such that* $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$. *Suppose that for each* $x \in [a,\infty)$, *the fuzzy integral* $\int_c^{\infty} f(x,t)dt$ *is convergent and moreover* $\int_a^{\infty} f(x,t)dx$ *as a function of* t *is convergent on* $[c,\infty)$. *Then*

$$\int_c^{\infty}\int_a^{\infty} f(x,t)dxdt = \int_a^{\infty}\int_a^{\infty} f(x,t)dtdx$$

Proof *Applying the theorem of Fubini-Tonelli [7] to these two functions* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$, *and use the conditions* (II.1)

Theorem II.20 *Suppose both,* $f(x,t)$ *and* $\partial_{x_{gH}} f(x,t)$, *are fuzzy continuous in* $[a,b] \times [c,\infty)$. *Suppose also that the integral converges for* $x \in \mathbb{R}$, *and the integral* $\int_c^{\infty} f(x,t)dt$ *converges uniformly on* $[a,b]$. *Then* F *is gH-differentiable on* $[a,b]$ *and*

$$F'_{gH}(x) = \int_c^{\infty} \partial_{x_{gH}} f(x,t)dt$$

Proof *The continuity of* $\partial_{x_{gH}} f(x,t)$ *on* $[a,b]$ *by the convergence domainee theorem of to* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$ *and use the condition* (II.1).

According to the theorem (II.8) we get

Theorem II.22 *Let* $f : [a,b] \to E^1$ *and* $g : [a,b] \to \mathbb{R}$ *are two differentiable functions (*f *is gH-differentiable), then*

$$\int_a^b f'_{gH}(x)g(x)dx = f(b)g(b) -_g f(a)g(a) -_g \int_a^b f(x)g'(x)dx$$

Remark II.23 *If* $f, g \in A^{E^1}$ *with* $\lim_{|x|\to\infty} f(x) = 0$, $\lim_{|x|\to\infty} g(x) = 0$ *then*

$$\int_{-\infty}^{\infty} f'_{gH}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)g'(x)dx$$

III. Fuzzy generalized Hukuhara partial differentiation

In this section $f : \mathbb{D} \subset \mathbb{R} \times \mathbb{R}^+ \to E^1$ is called the two variable fuzzy-valued function. The parametric representation of the fuzzy-valued function fis expressed by $f(x,t,\alpha) = \left[\underline{f}(x,t,\alpha), \overline{f}(x,t,\alpha)\right]$

Definition III.1 *[3] Let* $f : \mathbb{D} \subset \mathbb{R} \times \mathbb{R}^+ \to E^1$ *and* $(x_0,t_0) \in \mathbb{D}$. *Then first generalized Hukuhara partial derivative (*$[gH-p]$*-derivative for short) of* f *with respect to variables* x,t *are the functions* $\partial_{x_{gH}} f(x_0,t_0)$ *and* $\partial_{t_{gH}} f(x_0,t_0)$ *given by*

$$\lim_{h\to 0}\left\|\frac{f(x_0+h,t_0) -_g f(x_0,t_0)}{h} -_g \partial_{x_{gH}} f(x_0,t_0)\right\| = 0$$

and

$$\lim_{h\to 0}\left\|\frac{f(x_0,t_0+h) -_g f(x_0,t_0)}{h}, \partial_{x_{gH}} f(x_0,t_0)\right\| = 0$$

provided that $\partial_{x_{gH}} f(x_0,t_0), \partial_{t_{gH}} f(x_0,t_0) \in E^1$.

Definition III.2 *[3] Let* $f(x,t) : \mathbb{D} \to E^1$, $(x_0,t_0) \in \mathbb{D}$ *and* $\underline{f}(x,t;\alpha)$ *and* $\overline{f}(x,t;\alpha)$ *both partial differentiable w.r.t.* t *at* (x_0,t_0). *We say that*

- $f(x,t)$ *is* $[(i)-p]$*-differentiable w.r.t.* t *at* (x_0,t_0) *if*

$$\partial_{t_{i,gH}} f(x_0,t_0) = \left[\partial_t \underline{f}(x_0,t_0;\alpha), \partial_t \overline{f}(x_0,t_0;\alpha)\right] \tag{III.1}$$

$$\partial_{t_{ii,gH}} f(x_0,t_0) = \left[\partial_t \overline{f}(x_0,t_0;\alpha), \partial_t \underline{f}(x_0,t_0;\alpha)\right] \tag{III.2}$$

We inspired of the definition (II.11) we presented the following definition

Definition III.3 *$f : \mathbb{R} \times \mathbb{R}^+ R \to E^1$. We say that the function $t = h(x)$, is switching boundary for the differentiability of $f(x,t)$ with respect to t, if for all x belongs to domain of $h(x)$ and for all $t \in \mathbb{R}^+$, there exist points $t_0 < t_1 < t_2$ such that*

1) *at (x, t_1) (III.1) holds while (III.2) does not hold and at (x, t_2) (III.2) holds and (III.1) does not hold, or*
2) *at (x, t_1) (III.2) holds while (III.1) does not hold and at (x, t_2) (III.1) holds and (III.2) does not hold.*

Theorem III.4 *Consider $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$ and $u : \mathbb{R} \to E^1$ are fuzzy-valued functions such that $u(x;\alpha) = [\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$. Suppose that $h : \mathbb{R} \to \mathbb{R}$and $p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function w.r.t. t and*

$$\partial_t p(x,t) = \begin{cases} \partial_t p(x,t) \geq 0, & h_1(t) < x < h_2(t); \\ \partial_t p(x,t) < 0, & h_2(t) < x < h_3(t) \end{cases}$$

and $f(x,t) = p(x,t)u(x)$. Then $\partial_{t_{gH}} f(x,t)$ exists and

$$\partial_{t_{gH}} p(x,t) = \begin{cases} \partial_{t_{i,gH}} p(x,t) \geq 0, & h_1(t) < x < h_2(t); \\ \partial_{t_{ii,gH}} p(x,t) < 0, & h_2(t) < x < h_3(t) \end{cases}$$

In fact, the function $h_2(t)$ is switching boundary type 1 for differentiability of $f(x,t)$ with respect to t.

Proof *Since p is valued in $\mathbb{R}^+$ then we can set $f(x,t;\alpha) = p(x,t)[\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$, which implies that*

$$\partial_{t_{gH}} = \partial_t p(x,t)[\underline{u}(x;\alpha), \overline{u}(x;\alpha)]$$

.

If $h_1(t) < x < h_2(t)$ then

$$\partial_{t_{gH}} = [\partial_t p(x,t)\underline{u}(x;\alpha), \partial_t p(x,t)\overline{u}(x;\alpha)]$$

then $f(x,t)$ is [(i)-differentiable] by report at t. In the same if $h_2(t) < x < h_3(t)$ we get

$$\partial_{t_{gH}} = [\partial_t p(x,t)\overline{u}(x;\alpha), \partial_t p(x,t)\underline{u}(x;\alpha)]$$

thus $f(x,t)$ is [(ii)-differentiable] by report at t

IV. Generalized fuzzy fractional derivative

We present generalized fuzzy fractional derivative and their properties.

Definition IV.1 *[5] Let $f \in A^{E^1}([a,b])$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:*

$$I^q f(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^{q-1} f(s) ds,$$
$$a < s < t, \quad 0 < q < 1.$$

Definition IV.2 *[5] Let $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$ be a valued-fuzzy function. The fuzzy Riemann-Liouville integral of f is defined as following:*

$${}_{gH}D_t^q f(t,x;\alpha) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^q f'_{gH}(s) ds,$$
$$a < s < t, \quad 0 < q < 1$$

Also we say that f is $[(i) - gH]$-differentiable at t_0 if

$${}_{gH}D_t^q f(x,t;\alpha) = [D^q \underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$$

and f is $[(ii) - gH]$-differentiable at t_0 if

$${}_{gH}D_t^q f(x,t;\alpha) = [D^q \overline{f}(x,t;\alpha), \underline{f}(x,t;\alpha)]$$

Lemma IV.3 *Let $f \in A^{E^1}$ and $r \in (0,1)$,then*

1) *If f is $[(i) - gH]$-differentiable at t_0 then $D^r f$ is $[(i) - gH]$-differentiable at t_0.*
2) *If f is $[(ii) - gH]$-differentiable at t_0 then $D^r f$ is $[(ii) - gH]$-differentiable at t_0*

Proof *Note that*

$${}_{gH}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'_{gH}(s) ds$$

Since $\frac{1}{\Gamma(1-q)}(t-s)^{-q}$ is a nonegative quantity whenever $0 < t < s$.

Theorem IV.5 *Let $f \in A^{E^1}$ and $q \in (1,2)$,then*

$${}_{gH}D^q f(t) = {}_{gH}D^{q-1} f'_{gH}(t)$$

Proof *We set $f(t) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$and use lemma* (IV.3)
If f is [(i)-differentiable] then

$$f(t)' = [\underline{f}'(t;\alpha), \overline{f}'(t;\alpha)]$$

and

$$D^{q-1}f(t)' = [D^{q-1}\underline{f}'(t;\alpha), D^{q-1}\overline{f}'(t;\alpha)]$$

If f is [(i)-differentiable] then

$$f(t)' = [\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)]$$

and

$$D^{q-1}f(t)' = [D^{q-1}\overline{f}'(t;\alpha), D^{q-1}\underline{f}'(t;\alpha)]$$

Proposition IV.7 *Let $f : L^{E^1}$.*
If $D^{\gamma-1}f(t) = g(t)$, then $f(t) = f(0) + tf'_{gH}(0) + I^{\gamma-1}g(t)$

Proof *We set $f(t) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$ and $g(t) = [\underline{g}(t;\alpha), \overline{g}(t;\alpha)]$.*

1) *If f is [(i)-differentiable] by theorem (IV.5)*

$$\begin{aligned} D^{\gamma-1}f(t) &= [D^{\gamma-1}\underline{f}(t;\alpha), D^{\gamma-1}\overline{f}(t;\alpha)] \\ &= [\underline{g}(t;\alpha), \overline{g}(t;\alpha)] \end{aligned}$$

Which implies that

$$\begin{cases} D^{\gamma-1}\underline{f}(t;\alpha) = \underline{g}(t;\alpha) \\ D^{\gamma-1}\overline{f}(t;\alpha) = \overline{g}(t;\alpha) \end{cases}$$

By [9] we get

$$\begin{cases} \underline{f}(t;\alpha) = \underline{f}(0;\alpha) + t\underline{f}'(0;\alpha) + I^{\gamma-1}\underline{g}(t;\alpha) \\ \overline{f}(t;\alpha) = \overline{f}(0;\alpha) + t\overline{f}'(0;\alpha) + I^{\gamma-1}\overline{g}(t;\alpha) \end{cases}$$

in the same if f is [(ii)-differentiable] then

$$\begin{cases} \underline{f}(t;\alpha) = \underline{f}(0;\alpha) + t\overline{f}'(0;\alpha) + I^{\gamma-1}\underline{g}(t;\alpha) \\ \overline{f}(t;\alpha) = \overline{f}(0;\alpha) + t\underline{f}'(0;\alpha) + I^{\gamma-1}\overline{g}(t;\alpha) \end{cases}$$

Thus

$$f(t) = f(0) + tf'_{gH}(0) + I^{\gamma-1}g(t)$$

V. Fuzzy Fourier transform

In this section we discuss the Fourier transform in the fuzzy case

Lemma V.1 *If $f \in A^{E^1}$ then the map*

$$\begin{aligned} F: \quad & \mathbb{R} \longmapsto \mathbb{C}^1 \\ & \omega \to \int_{-\infty}^{\infty} f(x)e^{-i\omega w}dx \end{aligned}$$

is well defined

Proof *We have*

$$\left\| f(x)e^{-i\omega w} \right\| = \left\| f(x) \right\|$$

Since $f \in A^{E^1}$ then $f(x)e^{-i\omega w} \in A^{\mathbb{C}^1}$, which complet the proof.

Remark V.3 *In the same the map and under same assumption*

$$\begin{aligned} F: \quad & \mathbb{R} \longmapsto \mathbb{C}^1 \\ & \omega \to \int_{-\infty}^{\infty} f(x)e^{i\omega w}dx \end{aligned}$$

is well defined

By the previous lemma and remark we can give a definition of the fuzzy Fourier transform

Definition V.4 *Let $f : \mathbb{R} \to E^1$ a fuzzy-valued function. The fuzzy Fourier transform of f, denote $\mathcal{F}(f) : \mathbb{R} \to \mathbb{C}^1$, is given by*

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega w}dx = F(\omega)$$

Also the fuzzy inverse Fourier transform of $F(\omega)$ is given by

$$\mathcal{F}^{-1}(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\omega w}dx = f(x)$$

By the conditions (II.1) we have

Remark V.5 *Let $f \in A^{\mathbb{C}^1}$.*
If $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$, then we can denote

$$\mathcal{F}(f(x,t;\alpha)) = \left[\mathcal{F}\left(\underline{f}(x,t;\alpha)\right), \mathcal{F}\left(\overline{f}(x,t;\alpha)\right)\right]$$

with

$$[z_1, z_2] = [Re(z_1), Re(z_2)] \times [Im(z_1), Im(z_2)]$$

and

$$\mathcal{F}^{-1}(f(x,t;\alpha)) = \left[\mathcal{F}^{-1}\left(\underline{f}(x,t;\alpha)\right), \mathcal{F}^{-1}\left(\overline{f}(x,t;\alpha)\right)\right]$$

Using the conditions (II.1) and the linearity of Fourier transform on a "crisp" function we get for all $a, b > 0$

$$a\mathcal{F}(f(x,t;\alpha)) + b\mathcal{F}(g(x,t;\alpha)) = \mathcal{F}(af(x,t;\alpha) + bg(x,t;\alpha))$$

Theorem V.6 *Let $f \in A^{E^1}$ such that $\lim_{|x|\to\infty} f(x) = 0$. suppose that $f'_{gH} \in A^{E^1}$. Then*

$$\mathcal{F}\left(f'_{gH}(x)\right) = i\omega\mathcal{F}\left(f(x)\right)$$

Proof *Using theorem (II.22) we get*

$$\mathcal{F}\left(f'_{gH}(x)\right) = \frac{1}{\sqrt{2\pi}}\left[[f(x)e^{i\omega x}]_{-\infty}^{\infty} -_g (-i\omega)\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx\right]$$

Using the limite $\lim_{|x|\to\infty} f(x) = 0$ we get the result.

Corollary V.8 *If $f_{gH}^{(k)} \in A^{E^1}$ and $\lim_{|x|\to\infty} f^{(k)(x)} = 0$ for $k = 0, 1, 2$, then*

$$\mathcal{F}\left(f''_{gH}(x)\right) = -\omega^2\mathcal{F}\left(f(x)\right)$$

By the theorems (II.20) and (IV.5) we have

Theorem V.9

$$\mathcal{F}\left({}_{gH}D_t^{\gamma} f(x,t)\right) = {}_{gH}D_t^{\gamma}\mathcal{F}\left(f(x,t)\right)$$

VI. The solution of the fuzzy fractional wave equation

In this section consider the following problem

$$\begin{cases} {}_{gH}D_t^{\gamma}u(t,x) -_g c^2\frac{\partial^2}{\partial x^2}u(t,x) = 0 \\ 0 < x, t < 1,\ 0 < \gamma < 1 \\ u(0,x) = a(x), \\ \frac{\partial}{\partial t}u(0,x) = b(x) \end{cases} \tag{VI.1}$$

where a and b are belongs to A^{E^1},

Proposition VI.1 *the problem (VI.1) has a unique solution.*

Proof *Let $u(x,t)$ is fuzzy absolutely integrable, we define the fuzzy Fourier transform of $u(x,t)$ and its inverse by*

$$\mathcal{F}\left(u(x,t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega t}dx = U(\omega,t)$$

$$\mathcal{F}^{-1}\left(U(\omega,t)\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(\omega,t)e^{i\omega t}d\omega = u(x,t)$$

If $D_{t_{gH}}^{\gamma}u(x,t)$, $\partial_{x_{gH}}u(x,t)$ and $\partial_{xx_{gH}}u(x,t)$ are fuzzy absolutely integrable in $(-\infty,\infty)$ by using

$$\mathcal{F}\left({}_{gH}D_t^{\gamma}u(t,x)\right) -_g \mathcal{F}\left(c^2\frac{\partial^2}{\partial x^2}u(t,x)\right) = 0$$

It follows from the corollary (V.8) that

$$\mathcal{F}\left(c^2\frac{\partial^2}{\partial x^2}u(t,x)\right) = -c^2\omega^2 U(\omega,t)$$
$$\mathcal{F}\left({}_{gH}D_t^{\gamma}u(t,x)\right) = D_t^{\gamma}U(\omega,t)$$

We get

$${}_{gH}D_t^{\gamma}U(\omega,t) = -c^2U(\omega,t)$$

It follows that

$${}_{gH}D_t^{\gamma-1}U'_{gH}(\omega,t) = -c^2U(\omega,t)$$

Thus we have the following problem

$${}_{gH}D_t^{\gamma-1}U'_{gH}(\omega,t) = -c^2U(\omega,t) \tag{VI.2}$$
$$U(\omega,0) = \mathcal{F}\left(a(x)\right) \tag{VI.3}$$
$$\frac{\partial}{\partial t}U(\omega,0) = \mathcal{F}\left(b(x)\right) \tag{VI.4}$$

by lemma 3.2 [5] this problem has a unique solution given by

$$U(\omega,t) = U(\omega,0) + t\frac{\partial}{\partial t}U(\omega,0) -_g \frac{c^2}{\Gamma(\gamma-1)}\int_0^t\int_0^s (s-\tau)^{\gamma-2}U(\omega,\tau)d\tau ds$$

if u' is [(i)-differentiable], and

$$U(\omega,t) = U(\omega,0) + t\frac{\partial}{\partial t}U(\omega,0) + \frac{c^2}{\Gamma(\gamma-1)}\int_0^t\int_0^s (s-\tau)^{\gamma-2}U(\omega,\tau)d\tau ds$$

if u' is [(ii)-differentiable].
Which implies the existence and uniqueness of the solution of the problem (VI.2) and by the inverse of Fourier transform we get the existence and uniqueness of the solution of (VI.1).

VII. Case $\gamma = 2$

In this section we set

$$u(x,t;\alpha) = [\underline{u}(x,t;\alpha), \overline{u}(x,t;\alpha)]$$
$$a(x;\alpha) = [\underline{a}(x;\alpha), \overline{u}(x;\alpha)]$$
$$b(x;\alpha) = [\underline{b}(x;\alpha), \overline{b}(x;\alpha)]$$

If u' is [(i)-differentiable] then

$$\frac{\partial^2}{\partial t^2}\underline{u}(x,t;\alpha) = c^2\frac{\partial^2}{\partial x^2}\underline{u}(x,t;\alpha)$$
$$\frac{\partial^2}{\partial t^2}\overline{u}(x,t;\alpha) = c^2\frac{\partial^2}{\partial x^2}\overline{u}(x,t;\alpha)$$

which implies

$$\underline{u}(x,t;\alpha) = \underline{F}(x-ct;\alpha) + \underline{G}(x+ct;\alpha)$$
$$\overline{u}(x,t;\alpha) = \overline{F}(x-ct;\alpha) + \overline{G}(x+ct;\alpha)$$

where

$$\underline{a}(x;\alpha) = \underline{F}(x-ct;\alpha) + \underline{G}(x+ct;\alpha) \qquad \text{(VII.1)}$$

$$\overline{a}(x,t;\alpha) = \overline{F}(x-ct;\alpha) + \overline{G}(x+ct;\alpha) \qquad \text{(VII.2)}$$

and

$$\underline{b}(x;\alpha) = \underline{F'}(x-ct;\alpha) + \underline{G'}(x+ct;\alpha) \qquad \text{(VII.3)}$$

$$\overline{b}(x,t;\alpha) = \overline{F'}(x-ct;\alpha) + \overline{G'}(x+ct;\alpha) \qquad \text{(VII.4)}$$

By the conditions (II.1) the solution is given by

$$u(x,t) = F(x-ct) + G(x+ct)$$

where F and G are given by the above formula $(7.1) - (7.4)$.

VIII. Examples

In this section we will give some examples to illustrate the previous results.

Example VIII.1

$$\begin{cases} {}_{gH}D_t^{\frac{3}{2}} u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2} u(t,x) = 0 \\ 0 < x,t < 1,\ 0 < \gamma < 1 \\ u(0,x;\alpha) = [(1+\alpha)e^{-x^2}, (3-\alpha)e^{-x^2}], \\ \frac{\partial}{\partial t} u(0,x) = 0 \end{cases} \qquad \text{(VIII.1)}$$

the solution is given by $u(x,t) = \mathcal{F}^{-1}(U(\omega,t))$ *with*

$$U(\omega,t) = \left[\frac{\alpha+1}{\sqrt{2}} e^{-\omega^2}, \frac{-\alpha+3}{\sqrt{2}} e^{-\omega^2}\right] + \frac{c^2}{\Gamma(\frac{1}{2})} \int_0^t \int_0^s (s-\tau)^{-\frac{1}{4}} U(\omega,\tau) d\tau ds$$

Example VIII.2

$$\begin{cases} {}_{gH}D_t^2 u(t,x) -_g c^2 \frac{\partial^2}{\partial x^2} u(t,x) = 0 \\ 0 < x,t < 1,\ 0 < \gamma < 1 \\ u(x,0;\alpha) = [\alpha e^{-x^2}, (2-\alpha)e^{-x^2}], \\ \frac{\partial}{\partial t} u(x,0) = 0 \end{cases} \qquad \text{(VIII.2)}$$

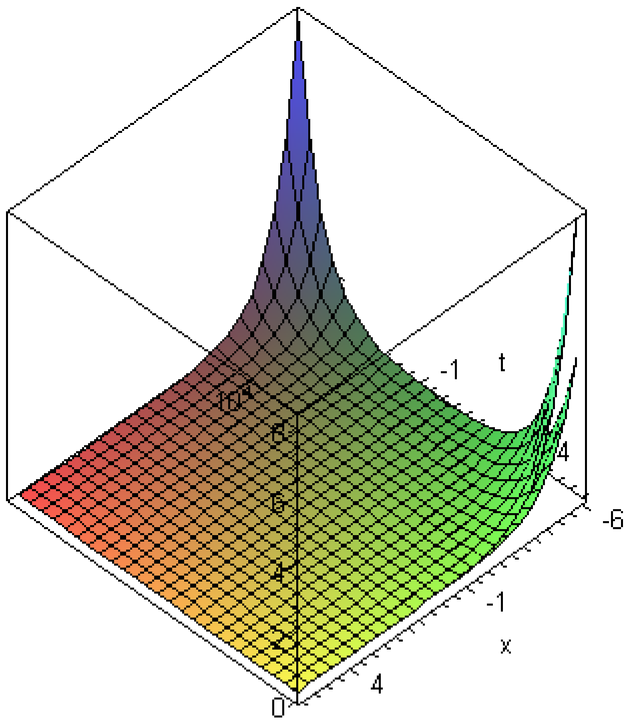

Fig. 1. Lower and upper branch of $u(x,t)$with $\alpha = 1$

the solution is given by

$$u(x,t) = [\alpha, 1 - \frac{\alpha}{2}] e^{-x} \cosh(ct)$$

IX. Conclusions

This study makes it possible to explain the wave phenomena with uncertainty in experimental data.

References

[1] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition, Fract. Calc. Appl. Anal. 15 (2012) 572-590.
[2] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear Anal. 72 (2010) 2859-2862.
[3] T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, On fuzzy solutions for heat equation based on generalized hukuhara differentiability, Fuzzy Sets Syst. 265 (2015) 1-23.
[4] G.A. Anastassiou, Fuzzy Mathematics: Approximation Theory. Studies in Fuzziness and Soft Computing, Springer, Berlin Heidelberg, 2010.

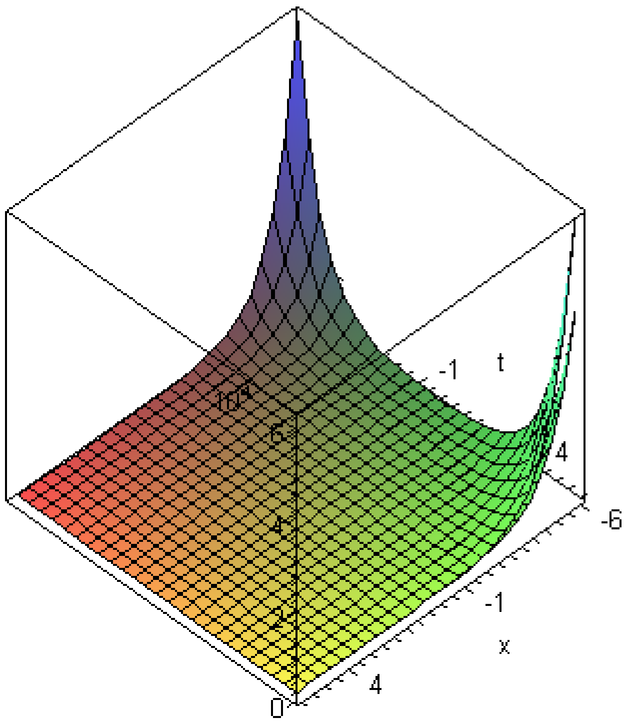

Fig. 2. Lower and upper branch of $u(x,t)$with $\alpha = 0.5$

[5] A. Armand, Z. Gouyandeh, Fuzzy fractional integro-differential equations under generalized Caputo differentiability. Annals of Fuzzy Mathematics and Informatics Volume 10, No. 5, (November 2015), pp. 789798.

[6] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, Fuzzy Sets Syst. 230(0) (2013) 119-141.

[7] Brezis.H. (2010) Functional Analysis, Sobolev Spaces and Partial Differential Equations,Springer.

[8] W. Congxin, M. Ming, Embedding problem of fuzzy number space: Part III, Fuzzy Sets Syst. 46(2) (1992) 281-286.

[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, New York, 2006

[10] J. Buckley, Fuzzy complex numbers, Fuzzy Sets Syst. 33(3) (1989) 333-345.

[11] L. Stefanini, B. Bede, Generalized hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal. 71(34) (2009) 1311-1328.

[12] H.C. Wu, The improper fuzzy Riemann integral and its numerical integration, Inf. Sci. 111(14) (1998) 109-137.

[13] Zadeh L. A., Fuzzy sets, Information and Control, 8, (1965), 338-353

Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability

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***Abstract*—In this paper, we establish the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for fuzzy integrodifferential equations under Caputo gH-differentiability by using the fixed point method.**

***Index Terms*—Fuzzy Ulam-Hyers-Rassias stability, Caputo fractional derivatives, fuzzy fractional integrodifferential equations, fixed point theory.**

I. Introduction

In this paper, we will propose fuzzy Ulam-Hyers-Rassias stability for the two kinds of fuzzy fractional integrodifferential equations of order $\alpha \in (0,1)$ with generalized Hukuhara derivative under form

$$\begin{cases} {}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}u(t) = f(t,u(t)) + \int_a^t g(t,s,u(s))ds, \ t \in [0,a], \\ u(0) = u_0 \in E^d. \end{cases} \quad (1)$$

Where ${}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}$ is the Caputo's generalized Hukuhara derivative, $f : [0,a] \times E^d \longrightarrow E^d$, is continuous on $[0,a]$ and $g : [0,a] \times [0,a] \times E^d \longrightarrow E$ is continuous on $[0,a] \times [0,a]$. We wish to mention that the theory of fuzzy fractional integral and differential equations have recently been the subject of important studies (see e.g [1]–[11]). In [12], Shen et al studied the Ulam stability problems of the first order linear fuzzy differential equations under some suitable conditions, and in [13], Diaz et al has introduced a fixed point theorem of the alternative for contractions on a generalized metric space, with which Shen et al in [14] proved the Ulam stability of fuzzy differential equations. Since the number of documents dealing with the stability of Ulam for fuzzy fractional integrodifferential equations (FFIEs) is rather limited compared to the number of publications concerning FFIEs, we decide to study by using the fixed point technique, the Ulam-Hyers-Rassias stability for FFIEs.

Our results are inspired by the one in [15] where the fuzzy Ulam-Hyers-Rassias stability of FFIEs is studied. The rest of this paper is organized as follows: In section 2, we recall some notations of the fuzzy number space, the fixed point theorem and the basic notations of the Riemann-Liouville fractional integral and Caputo fractional derivative for fuzzy functions. The Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations are discussed in Sections 3.

II. Preliminaries

In this section, we introduce some definitions, theorems and lemmas which are used in this paper. For more details, we can see papers [3] [9] [12].

Definition 2.1: A function $d : \mathbb{X} \times \mathbb{X} \longrightarrow [0,+\infty)$ is called a generalized metric on $\mathbb{X}$ if and only if d satisfies:

(1) $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{X}$,

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in \mathbb{X}$.

Theorem 2.2: $(Banach)$ Let $d : \mathbb{X} \times \mathbb{X} \longrightarrow [0,+\infty)$ be a generalized metric on $\mathbb{X}$ and $(\mathbb{X}, d)$ is a generalized complete metric space. Assume that $T : \mathbb{X} \longrightarrow \mathbb{X}$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer n such that $d(T^{n+1}x, T^n x) < \infty$ for some $x \in \mathbb{X}$, then the following are true:

(i) the sequence $T^n x$ converges to a fixed point x^* of T,

(ii) x^* is the unique fixed point of T in $\mathbb{X}^* = \{y \in \mathbb{X} \mid d(T^n x, y) < \infty\}$,

(iii) if $y \in \mathbb{X}^*$, then we have $d(y, x^*) \leq \frac{1}{1-L} d(Ty, y)$.

Lemma 2.3: Let $\varphi : J \longrightarrow [0,+\infty)$ be a continuous function. We define the set

$$\mathbb{X} := \{x : J \longrightarrow \mathbb{R}_{\mathcal{F}} \mid x \ is \ continuous \ function \ on \ J\},$$

where $\mathbb{R}_{\mathcal{F}}$ is the space of fuzzy sets, equipped with the metric $d(x,y) = \inf\{\eta \in [0,+\infty) \cup \{+\infty\} \mid D(x(t), y(t)) \leq \eta\varphi(t), \ \forall t \in J\}$.

Then, $(\mathbb{X}, d)$ is a complete generalized metric space.

Let $K_c(\mathbb{R}^d)$ denote the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. The addition and scalar multilplication in $K_c(\mathbb{R}^d)$ are defined as usual i.e, for $A, B \in K_c(\mathbb{R}^d)$ and $\lambda\mathbb{R}$,

$$A + B = \{a + b \mid a \in A, b \in B\}, \qquad \lambda A = \{\lambda a \mid a \in A\}$$

Let E^d denote the set of fuzzy subsets of the real axis, if $\omega : \mathbb{R}^d \longrightarrow [0,1]$, satisfying the following properties:
(i) ω is normal, that is, there exists $z_0 \in \mathbb{R}^d$ such that $\omega(z_0) = 1$,

(ii) ω is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$\omega(\lambda z_1 + (1-\lambda) z_2) \geq min\{\omega(z_1), \omega(z_2)\},\ for\ any\ z_1, z_2 \in \mathbb{R}^d,$$

(iii) ω is upper semicontinous on $\mathbb{R}^d$,

(iv) $[\omega]^0 = cl\{z \in \mathbb{R}^d : \omega(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, |\ .\ |)$.
Then E^d is called the space of fuzzy number. For $r \in (0,1]$, we denote $[\omega]^r = \{z \in \mathbb{R}^d \mid \omega(z) \geq r\}$ and $[\omega]^0 = \{z \in \mathbb{R}^d \mid \omega(z) > 0\}$. From the conditions (i) to (iv), it follows that the $r-level$ set of ω, $[\omega]^r$, is a nonempty compact interval, for all $r \in [0,1]$ and any $\omega \in E$.

The notation $[\omega]^r = [\underline{\omega}(r), \overline{\omega}(r)]$, denotes explicitly the $r-level$ set of ω, for $r \in [0,1]$. We refer to $\underline{\omega}$ and $\overline{\omega}$ as the lower and upper branches of ω, respectively. For $\omega \in E^d$, we define the lengh of the $r-level$ set of ω as $len([\omega]^r) = \overline{\omega}(r) - \underline{\omega}(r)$. For addition and scalar multiplication in fuzzy set space E^d, we have $[\omega_1 + \omega_2]^r = [\omega_1]^r + [\omega_2]^r$, $[\lambda\omega]^r = \lambda[\omega]^r$.

The Hausdorff distance between fuzzy numbers is given by

$$D_0[\omega_1, \omega_2] = \sup_{0 \leq r \leq 1} \{|\ \underline{\omega}_1(r) - \underline{\omega}_2(r)\ |, |\ \overline{\omega}_1(r) - \overline{\omega}_2(r)\ |\}.$$

The metric space (E^d, D_0) is complet metric space and the following properties of the metric D_0 are valid.

$$D_0[\omega_1 + \omega_3, \omega_2 + \omega_3] = D_0[\omega_1, \omega_2],$$

$$D_0[\lambda\omega_1, \lambda\omega_2] = |\ \lambda\ |\ D_0[\omega_1, \omega_2],$$

$$D_0[\omega_1, \omega_2] \leq D_0[\omega_1, \omega_3] + D_0[\omega_3, \omega_2],$$

for all $\omega_1, \omega_2, \omega_3 \in E^d$ and $\lambda \in \mathbb{R}^d$. Let $\omega_1, \omega_2 \in E^d$, if there exists $\omega_3 \in E^d$ such that $\omega_1 = \omega_2 + \omega_3$ then ω_3 is called the H-difference of ω_1, ω_2. We denote the ω_3 by $\omega_1 \ominus \omega_2$. Let us remark that $\omega_1 \ominus \omega_2 \neq \omega_1 + (-1)\omega_2$.

Definition 2.4: The generalized Hukuhara difference of two fuzzy numbers $\omega_1, \omega_2 \in E^d$ (gH-difference for short) is defined as follows:

$$\omega_1 \ominus_{gH} \omega_2 = \omega_3 \Leftrightarrow \begin{cases} (i)\ \omega_1 = \omega_2 + \omega_3, \\ or\ (ii)\ \omega_2 = \omega_1 + (-1)\omega_3. \end{cases}$$

Let $[0,a]$ be a compact interval in $\mathbb{R}^+$. Denote by $diam[u(t)]^r$ the diameter of fuzzy set u, for $t \in [0,a]$. A function $u : [0,a] \longrightarrow E^d$ is called ω-increasing (ω-decreasing) on $[0,a]$ if for every $r \in [0,1]$ the function $t \longmapsto diam[u(t)]^r$ is nondecreasing (nonincreasing) on $[0,a]$. If u is ω-increasing or ω-decreasing on $[0,a]$, then we say that u is ω-monotone on $[0,a]$.

Definition 2.5:

Let $t \in (a,b)$ and h such that $t+h \in (a,b)$, then the generalized Hukuhara derivative of fuzzy-valued function $x : (a,b) \longrightarrow E^d$ at t is defined as

$$D_{gH}x(t) = \lim_{h \longrightarrow 0} \frac{x(t+h) \ominus_{gH} x(t)}{h}.$$

If $D_{gH}x(t) \in E^d$ satisfying last inequality, we say that x is generalized Hukuhara differentiable (gH-differentiable for short) at t.

Definition 2.6: Let $x : [a,b] \longrightarrow E^d$, the fuzzy Rieman-Liouville integral of fuzzy-valued function x is defined as follows:

$$(\mathcal{J}^\alpha_{a^+} x)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds.$$

For $a \leq t$, and $0 < \alpha \leq 1$. For =$\alpha = 1$, we set $\mathcal{J}^1_a = I$, the identity operator.

Definition 2.7: Let $D_{gH} \in C([a,b], E^d) \cap L([a,b], E^d)$. The fuzzy gH-fractional Caputo diffentiability of fuzzy-valued function x $([gH]^C_a - differentiable\ for\ short)$ is defined as following:
${}^C_{gH}\mathcal{D}^\alpha_{a^+} x(t) = \mathcal{J}^{1-\alpha}_{a^+}(D_{gH}x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} (D_{gH}x)(s) ds$,
where $0 < \alpha \leq 1$, $t > a$.

Lemma 2.8: Suppose that $x : [a,b] \longrightarrow E^d$ be a fuzzy function and $D_{gH}x(t) \in C([a,b], E^d) \cap L([a,b], E^d)$. Then

$$\mathcal{J}^\alpha_{a^+}({}^C_{gH}\mathcal{D}^\alpha_{a^+} x)(t) = x(t) \ominus_{gH} x(a).$$

Lemma 2.9: Let $u : [0,a] \longrightarrow E^d$ be a continuous function on $[0,a]$ and let $\alpha \in (0,1)$, then the FFIE (1) is equivalent to the following integral equation:

(1) If u is ω-increasing on $[0,a]$, then

$$u(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s,u(s)) + \int_a^s g(s,r,u(r)) dr) ds, \tag{2}$$

(2) If u is ω-decreasing on $[0,a]$, then

$$u(t) = \varphi(0) \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (f(s,u(s)) + \int_a^s g(s,r,u(r)) dr) ds. \tag{3}$$

III. Main results

In the sequel, our aim score is to present the results for the existence and the stability of the problem (1). The methods to solve these problems are quite similar. However, since the conditions for the existence of solutions of fuzzy fractional integrodifferential equations (2) and (3) are dissimilar, we shall present the two kinds (2) and (3) in two separate subsections.

A. Fuzzy Ulam-Hyers-Rassias stability for FFIEs (2)

Firstly, we present the definitions of fuzzy Ulam-Hyers stability and fuzzy Ulam-Hyers-Rassias stability.

Definition 3.1: We say that the problem (2) is fuzzy Ulam-Hyers stable, if there exists a constant $K_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0,a], E^d)$ of the following inequality
$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}v(t), f(t,v(t)) + \int_a^t g(t,s,v(s))ds\right] \leq \varepsilon, \forall t \in [0,a]$,
then, there exists a solution $u \in C^1([0,a], E^d)$ of problem (2) with

$$D\left[v(t), u(t)\right] \leq K_f \varepsilon,$$

for all $t \in [0,a]$. We call K_f a Ulam-Hyers stability constant of (2).

Definition 3.2: We say that the problem (2) is fuzzy Ulam-Hyers-Rassias stable, if there exists a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1([0,a], E^d)$ of the following inequality
$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}v(t), f(t,v(t)) + \int_a^t g(t,s,v(s))ds\right] \leq \varphi(t), \forall t \in [0,a]$,
then, there exists a solution $u \in C^1([0,a], E^d)$ of problem (2) with

$$D\left[v(t), u(t)\right] \leq C_f \varphi(t),$$

for all $t \in [0,a]$ and for some nonnegative function φ defined on $[0,a]$.

Remark 3.3: We observe that definition 3.2 $\Rightarrow$ definition 3.1.

In the following, we shall prove that the FFIEs (2) is fuzzy Ulam-Hyers-Rassias stable on bounded interval by the fixed point theorem.

Theorem 3.4: Assume that $f : [0,a] \times E^d \longrightarrow E^d$ and $g : [0,a] \times [0,a] \times E^d \longrightarrow E^d$ are continuous functions satisfying the following conditions:

(i) There exists a constant $L_{fg} > 0$ such that:

$$\max\left\{D[f(t,u), f(t,v)]; D[g(t,s,u), g(t,s,v)]\right\} \leq L_{fg} D[u,v], \tag{4}$$

for all each $(t,s,u), (t,s,v) \in [0,a] \times [0,a] \times E^d$.

(ii) There exists a constant $K, C > 0$ such that $0 < L_{fg}K(1+C) < 1$ and let $\varphi : [0,a] \longrightarrow [0,\infty)$ be a continuous function and increasing on $[0,a]$ with:

$$\int_a^t \varphi(s)ds \leq C.\varphi(t), \qquad \forall t \in [0,a], \tag{5}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds \leq K\varphi(t), \qquad \forall t \in [0,a], \tag{6}$$

If a continuously ω-increasing function $u : [0,a] \longrightarrow E^d$ satisfies the following inequality

$$D\left[{}^{C}_{gH}\mathcal{D}^{\alpha}_{a^+}u(t), f(t,u(t)) + \int_a^t g(t,s,u(s))ds\right] \leq \varphi(t), \tag{7}$$

for any $t \in [0,a]$, then there exists a unique $\tilde{u} : [0,a] \longrightarrow E^d$ of (2.2) such that

$$\tilde{u}(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,\tilde{u}(s)) + \int_a^s g(s,r,\tilde{u}(r))dr)ds, \tag{8}$$

and

$$d[\tilde{u}(t), u(t)] \leq \frac{1}{1 - L_{fg}K(1+C)}, \qquad \forall t \in [0,a]. \tag{9}$$

Proof:

Let us consider the space of all continuous fuzzy function $u : [0,a] \longrightarrow E^d$ by

$$\mathbb{X} = \{u : [0,a] \longrightarrow E^d \mid u \textit{ is continuous on } [0,a]\},$$

equipped by the metric

$d(u,v) = \inf\{C \in [0,+\infty) \cup \{+\infty\} \mid D[u(t), v(t)] \leq C\varphi(t)\}, \ \forall t \in [0,a]$.

By lemma 2.3, we observe that $(\mathbb{X}, d)$ is also a complete generalized metric space. We define an operator $Q : \mathbb{X} \longrightarrow \mathbb{X}$ by

$$(Qu)(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s)) + \int_a^s g(s,r,u(r))dr)ds, \qquad \forall t \in [0,a]. \tag{10}$$

Because f and g are a continuous fuzzy functions, the right hand side of (10) is also continuous on $[0,a]$. This yields that Qu is continuous on $[0,a]$. So, the operator Q is well-defined. To apply theorem 2.2 in the proof of this theorem, we need the operator Q to be strict contractive on $\mathbb{X}$. For any $u, v \in \mathbb{X}$ and let $C_{uv} \in [0,+\infty) \cup \{+\infty\}$ such that

$$d(u,v) \leq C_{uv}, \qquad \forall t \in [0,a].$$

Then, by the definition of d, we have

$$D[u(t), v(t)] \leq C_{uv}\varphi(t), \qquad \forall t \in [0,a]. \tag{11}$$

From the definition of the operator Q and assumption (4)-

(6), we have the following estimation

$$D[(Qu)(t),(Qv)(t)] = D[u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s)) + \int_a^s g(s,r,u(r))dr)ds, u_0 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,v(s)) + \int_a^s g(s,r,v(r))dr)ds],$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D[f(s,u(s)), f(s,v(s))]ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s D[g(s,r,u(r)), g(s,r,v(r))]dr)ds,$$
$$\leq \frac{L_{fg}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} D[u(s),v(s)]ds + \frac{L_{fg}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s D[u(r),v(r)]dr)ds,$$
$$\leq \frac{L_{fg}C_{uv}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varphi(s)ds + \frac{L_{fg}C_{uv}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(\int_a^s \varphi(r)dr)ds,$$
$$\leq L_{fg}C_{uv}K\varphi(t) + L_{fg}C_{uv}CK\varphi(t) = L_{fg}K(1+C)C_{uv}\varphi(t).$$

Hence

$$D[(Qu)(t),(Qv)(t)] \leq L_{fg}K(1+C)C_{uv}\varphi(t). \tag{12}$$

So, by the definition of metric d, we get

$$d(Qu,Qv) \leq L_{fg}K(1+C)d(u,v), \quad for\ all\ u,v \in E^d.$$

Where $0 < L_{fg}K(1+C) < 1$, hence the operator Q is strictly contractive mapping on $\mathbb{X}$.
For an arbitrary $\omega \in \mathbb{X}$ and from the definition of $\mathbb{X}$ and Q, it follows that there exists a constant $0 < C_\omega < \infty$ such that:

$$D[(Q\omega)(t),\omega(t)] = D[u_0 + \tfrac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,\omega(s)) + \int_a^s g(s,r,\omega(r))dr)ds, \omega(t)] \leq C_\omega\varphi(t),$$

for any $t \in [0,a]$, since f,g and ω are bounded on $[0,a]$, and the minimum of $\varphi(t) > 0$ on $t \in [0,a]$. Then, we infer that $d(Q\omega,\omega) \leq C_\omega < \infty$. Therefore, according to (i) and (ii) of theorem 2.2, there exists a continuously function $\tilde{u} : [0,a] \longrightarrow E^d$ such that $Q^n\omega \longrightarrow \tilde{u}$ in the space $(\mathbb{X},d)$ as $n \longrightarrow \infty$ and $Q\tilde{u} = \tilde{u}$, that $\tilde{u}$ satisfies the problem (8) for any $t \in [0,a]$.
Now, we shall confirm that $\{u \in \mathbb{X} \mid d(\omega,u) < \infty\} = \mathbb{X}^*$. For an arbitrary $u \in E^d$, since u and ω are bounded on $[0,a]$ and $\min_{t\in[0,a]}\varphi(t) > 0$, there exists a constant $0 < C_u < \infty$ such that $D[\omega(t),u(t)] \leq C_u\varphi(t)$ for any $t \in [0,a]$. Therefore, we have $d(\omega,u) < \infty$ for any $u \in E^d$, that is $\{u \in \mathbb{X} \mid d(\omega,u) < \infty\} = \mathbb{X}^*$. By theorem 2.2-$(ii)$, we conclude that $\tilde{u}$ is the unique fixed point of Q on $\mathbb{X}$.
On the other hand, from the inequality (7) it follows that

$$d(u,Qu) \leq 1. \tag{13}$$

Finally, by theorem $2.2-(iii)$ and from the estimation (13), it implies that

$$d(\tilde{u}(t),u(t)) \leq \frac{d(u,Qu)}{1-L_{fg}K(1+C)} \leq \frac{1}{1-L_{fg}K(1+C)},$$

which means the estimation (9) holds true for any $t \in [0,a]$. This completes the proof. □

B. Fuzzy Ulam-Hyers-Rassias stability for FFIEs (3)

Theorem 3.5: Suppose that the functions f,g and φ satisfy all conditions as in theorem 3.4. Assume that for each $t \in [0,a]$ and for each continuous fuzzy function $z : [0,a] \longrightarrow E^d$, if the Hukuhara difference $z(0) \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(f(s,z(s)) + \int_a^s g(s,r,z(r))dr\right)ds$, exists and a continuously ω-nonincreasing function $v : [0,a] \longrightarrow E^d$ satisfies

$$D[v(t), v_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,v(s)) + \int_a^s g(s,r,v(r))dr)ds] \leq \varphi(t), \tag{14}$$

for any $t \in [0,a]$, where $v_0 = u_0$, then there exists a unique solution $\hat{u} : [0,a] \longrightarrow E^d$ of the problem (3) which satisfies

$$\hat{u}(t) = u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,\hat{u}(s)) + \int_a^s g(s,r,\hat{u}(r))dr)ds, \tag{15}$$

and

$$d[\hat{u}(t),v(t)] \leq \frac{1}{1-L_{fg}K(1+C)}, \tag{16}$$

for any $t \in [0,a]$.

Proof:
We consider the complete generalized space $(\mathbb{X},d)$ defined as in the proof of theorem 2. Define the operator $P : \mathbb{X} \longrightarrow \mathbb{X}$ as follows:

$$(Pu)(t) = u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(f(s,u(s)) + \int_a^s g(s,r,u(r))dr)ds, \quad t \in [0,a]. \tag{17}$$

Since the function f and g is continuous on $[0,a]$ and the Hukuhara difference $u_0 \ominus \frac{(-1)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(f(s,u(s)) + \int_a^s g(s,r,u(r))dr\right)ds$ exists, similary to theorem 1, it follows that Pu is well-defined on $[0,a]$ or Pu is continuous on $[0,a]$. Now, we observe that the operator P is strictly contractive on $\mathbb{X}$. Indeed, for any $u,v \in \mathbb{X}$ and let $C_{uv} \in [0,+\infty) \cup \{+\infty\}$ be an arbitrary constant with $d(u,v) \leq C_{uv}$ for $t \in [0,a]$, that is, let us assume that

$$D[u(t),v(t)] \leq C_{uv}\varphi(t), \tag{18}$$

for $t \in [0, a]$. Furthermore, from (17), (18) and by the Lipschitz condition of f and g, we have the following estimation:

$$D[(Pu)(t), (Pv)(t)] = D[u_0 \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(f(s, u(s))$$
$$+ \int_a^s g(s, r, u(r))dr)ds, u_0 \ominus \frac{(-1)}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(f(s, v(s))$$
$$+ \int_a^s g(s, r, v(r))dr)ds],$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D[f(s, u(s)), f(s, v(s))]ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (\int_a^s D[g(s, r, u(r)), g(s, r, v(r))]dr)ds,$$
$$\leq \frac{L_{fg}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D[u(s), v(s)]ds$$
$$+ \frac{L_{fg}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (\int_a^s D[u(r), v(r)]dr)ds,$$
$$\leq L_{fg} C_{uv} K \varphi(t) + L_{fg} C_{uv} C K \varphi(t)$$
$$= L_{fg} K(1 + C) C_{uv} \varphi(t).$$

Hence

$$D[(Pu)(t), (Pv)(t)] \leq L_{fg} K(1 + C) C_{uv} \varphi(t). \tag{19}$$

This means that $d(Pu, Pv) \leq L_{fg} K(1 + C) d(u, v)$. Hence, the operator P is a strictly contractive mapping on $\mathbb{X}$ by the assumption $0 < L_{fg} K(1 + C) < 1$. Simalar to the theorem 3.4, we can show that for each $\omega \in \mathbb{X}$ satisfies $d(P\omega, \omega) < \infty$. Hence, by theorem 1, it implies that there exists a continuously function $\hat{u} : [0, a] \longrightarrow E^d$ such that $P^n\omega \longrightarrow \hat{u}$ in $(\mathbb{X}, d)$ as $n \longrightarrow \infty$, and such that $P\hat{u} = \hat{u}$, that is $\hat{u}$ satisfies (4.15) for $t \in [0, a]$. Similar to the proof of theorem 3.4, we observe that there exists a constant $C_\omega > 0$ such that $D[\omega(t), u(t)] \leq C_\omega$, for any $t \in [0, a]$. This means that $d(\omega, u) < \infty$ for each $u \in E^d$, or equivalently, $\{u \in \mathbb{X} \mid d(\omega, u) < \infty\} = \mathbb{X}^*$. Furthermore, by theorem 2.2, we imply that $\hat{u}$ is a unique continuous function which satisfies (15).
Moreover, by theorem 2.2, we also obtain

$$d(\hat{u}(t), u(t)) \leq \frac{d(u, Pu)}{1 - L_{fg} K(1 + C)} \leq \frac{1}{1 - L_{fg} K(1 + C)},$$

which means the estimation (16) holds true for any $t \in [0, a]$. This completes the proof. □

IV. CONCLUSION

In this study, we are studied the Ulam-Hyers-Rassias stability for fuzzy intergodifferential equation via the fixed point technique. This result can be used to study fractional fuzzy differential equations with other types of derivative concepts in fuzzy setting, for example, Riemann-Liouville and Hadamard generalized Hukuhara differentiability.

REFERENCES

[1] T. V. An, H. Vu, N. V. Hoa, A new technique to solve the initial value problems for fractional fuzzy delay differential equations, Advances in Difference Equations, 181 (2017),

[2] T. V. An, H. Vu, N. V. Hoa, Applications of contractive-like mapping principles to interval-valued fractional integro-differential equations, Journal of Fixed Point Theory and Applications, 19(4) (2017), 2577-2599.

[3] T. V. An, H. Vu, N. V. Hoa, Hadamard-type fractional calculus for fuzzy functions and existence theory for fuzzy fractional functional integro-differential equations, Journal of Intelligent and Fuzzy Systems, 36(4) (2019), 3591-3605.

[4] N. V. Hoa, Fuzzy fractional functional differential equations under Caputo gH-differentiability, Communications in Nonlinear Science and Numerical Simulation, 22(1) (2015), 1134-1157.

[5] N. V. Hoa, Fuzzy fractional functional integral and differential equations, Fuzzy Sets and Systems, 280 (2015), 58-90.

[6] N. V. Hoa, V. Lupulescu, D. Oregan, Solving interval-valued fractional initial value problems under Caputo gH-fractional differentiability, Fuzzy Sets and Systems, 309 (2017), 1-34.

[7] N. V. Hoa, H. Vu, T. Minh Duc, Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach, Fuzzy Sets and Systems, 375(15) (2019), 70-99.

[8] H. V. Long, N. T. K. Son, N. V. Hoa, Fuzzy fractional partial differential equationd in partially ordered metric spaces, Iranian Journal of Fuzzy Systems, 14(2) (2017), 107-126.

[9] M. T. Malinowski, Random fuzzy fractional integral equations-theoretical foundations, Fuzzy Sets and Systems, 265 (2015), 39-62.

[10] N. Ngoc Phung, B. Quoc Ta, H. Vu, Hyers-Ulam stability and Hyers-Ulam-Rassias stability for fuzzy integrodifferential equation, Complexity, 2019 (2019),

[11] H. Vu, V. Lupulescu, N. Van Hoa, Existence of extremal solutions to interval-valued delay fractional differential equations via monotone iterative technique, Journal of Intelligent and Fuzzy Systems, 34(4) (2018), 2177-2195.

[12] Y. Shen, On the Ulam stability of
rst order linear fuzzy differential equations under generalized differentiability, Fuzzy Sets and Systems, 280(C) (2015), 27-57.

[13] J. B. Diaz, B. Margolis, A
fixed point theorem of the alternative, for contractions on a generalized complete metric space, Bulletin of the American Mathematical Society, 74(2) (1968), 305-309.

[14] Y. Shen, On the Ulam stability of first order linear fuzzy differential equations under generalized differentiability, Fuzzy Sets and Systems, 280(C) (2015), 27-57.

[15] H. Vu, J. M. Rassias and N. Van Hoa, Ulam-Hyers-Rassias stability for fuzzy fractional integral equations, Iranian Journal of Fuzzy Systems Volume 17, Number 2, (2020), pp. 17-27.

Eco-taxation as an instrument to fight against climate change

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***Abstract*— Taxation is one of the instruments for changing the behavior of agents, either by encouraging or discouraging certain behavior's considered good or bad, for instance certain investments with negative or positive impacts that arouse government's concerns.**

Eco-taxation seeks to change the behavior of agents by using the instrument of taxation to discourage, especially, negligent behavior leading to more pollution and climate change. Eco-taxation also aims to implement the polluter-pays principle, that is, to make the polluter bear the cost of his environmental damage, as well as allowing the State to generate tax revenues that can be directed towards financing appropriate policies for a transition towards a more environmentally friendly economy. Eco-taxation can thus be used to limit the production and consumption of goods and services that are harmful to the environment.

This research will compare all the tax levies existing in Europe and in Africa (the case of Morocco), as these instruments are designed to meet the challenges of climate change, and to show how coherent eco-taxation contribute effectively to changing the behavior of all economic actors.

Keywords— Eco-taxation, environmental taxation, ecological taxation, climate change.

I. Introduction

Earth's climate is regulated by the ability of the atmosphere to partially retain the energy reflected from the earth. This physical phenomenon is called the greenhouse effect, because it is similar to that encountered in a glass greenhouse. It is a natural phenomenon essential to the development of life on earth; in its absence, there would be no liquid water, because the average temperature on earth would reach degrees much lower than the current temperature.

These problems of economic changes, posed by the environment were partly behind the creation of certain taxes and levies, because companies when they buy, sell or fix the price of products, do not directly integrate the cost of the damages that 'they cause the environment, and the future scarcity of energies and raw materials.

Eco-taxation aims to integrate the environmental cost it causes into the cost price; it is thus a means of changing the behavior of economic agents in a way that is favorable to the environment.

The use of Eco-taxation is also justified by the "polluter pays" principle, that is to say, the polluter participates in the financing of measures to prevent, reduce and fight against pollution, via their tax contribution.

In this contribution, we will analyze all the levies aimed at combating existing climate change in Morocco and in certain European countries, our problematic will be as follows: to what extent does eco-taxation effectively contribute to change the behavior of economic players?

We will try to answer this problem through three axes:
Axis I: Theoretical overview on Eco-taxation;
Axis II: Eco-Taxation in Europe;
Axis III: Eco-Taxation in Morocco.

Axis I: Theoretical overview on Eco-taxation.

Eco-taxation aims to solve the following problems:

- Fight against global warming;
- Reduction of pollution;
- Rational use of resources;
- Preservation of natural environments of biodiversity.

Indeed, Eco-taxation, is a mode of production, a product, or a service damaging the environment ("evils"), makes it possible to limit the attacks only in the interest of allowing the public authorities to finance the damages of the public expenses.

The main difference between taxation and Eco-taxation stems is its objective, where taxation is defined as the means for the State to collect the resources which will enable it to finance these expenses, or more generally public goods (education , defense, health, etc.) or to ensure a certain redistribution of income; while Eco-taxation aims to modify the behavior of economic agents in order to prevent them from making decisions that they might regret in the future, or thus to limit consumption to protection of the Environment .

Several terms refer to the concept of Eco-taxation, such as: Green Taxation, Environmental Taxation, Ecological Taxation, Energy Taxation, and Eco Tax...

The notion of Eco-taxation is a challenge, which has given rise to numerous debates on the most relevant perimeter to be retained, which bring us to the definition of the OECD [1], "all taxes and fees whose base is constituted by a pollutant or, more generally, by a product or a service which deteriorates the environment or which results in a levy on natural resources ".

According to this definition, Eco-taxation goes beyond the sole taxation explicitly designed to fight against pollution, but incorporates another main purpose is the financing of public services; also it has the effect of helping to limit pollution.

The tax council in France in 2005[2]; announces that "The situation can be considered as paradoxical: the most important environmental effects are due to taxes and fees for services rendered, created long before the emergence of public policies in favor of the environment; fiscal measures directly inspired by environmental concerns have only a limited effect, whether it concerns the various components of the tax on polluting activities (TGAP [3]) or derogatory tax measures "

This announcement is considered as a broad approach, eco-taxation is an important instrument to face the challenges of climate change, by using only energy taxation, the objective is to change the behavior of all agents for a transition to an economy low in carbon and more environmentally friendly.

This approach was confirmed by Thierry Wahl, Inspector General of Finance and responsible for a report on the subject, "the most successful expression of the polluter pays principle". It is intended to encourage virtuous behavior in environmental matters and to deter bad behavior. It can take several forms: tax, royalties, tax credit, exemption or even direct aid. [4]

Environmental taxation also allows the internalization of external costs, that is to say to pass on in the price of goods and services, certain environmental costs which are currently unduly supported by future generations, the objective is to push the agents to make financially and environmentally sound decisions.

The externality or external effects appearing when the decisions of an economic agent affect the well-being of other agents, involuntarily, despite the absence of any market transaction between them.

The external effects can be positive (beneficial influence) or negative (deterioration of the situation). Pollution constitutes a negative externality therefore, to get polluters to take into account the external cost of their activity, the regulatory world proposed by Arthur C. Pigou [5] consists of the implementation of a tax called " Pigovienne Tax ", whose unit tax must be equal to the marginal damage caused by polluting emissions at their optimal level (that is to say the level which maximizes social welfare, the marginal damage suffered by the victims is equal to marginal cost of cleaning up the polluting sector) it provides the price signal which ensures the internalization of externalities.

At this level, all polluters want to minimize their costs either by motivating innovation to seek less polluting solutions to reduce their production costs or by offering less polluting products by taking advantage of the opportunities provided by environmental regulations.

In general, the increase in the price of the polluting good or service due to the environmental tax results in an increase in its price compared to other goods and services. This increase in prices encourages consumers and buyers to change their decisions vis-à-vis this type of polluting goods and service, by choosing other non-polluting ones, this decision change is desired precisely because it comes from correct the behavior of agents against the environment resulting from climate change.

As a conclusion, environmental taxation aims to integrate additional costs in the form of environmental taxes (called "externalities") into the cost borne by each of the economic parties. in addition to the regulatory approach, and stimulates innovation in the medium term respects the polluter pays principle, defined by the Organization for Economic Cooperation and Development (OECD) in 1972, which assumes that the costs resulting from pollution prevention, reduction and control measures must be borne by the polluter.

*Axis II: Eco-Taxation in Europe_ case of France_*

Eco-taxation is the set of taxes, fees and charges that are imposed on polluting taxpayers, more generally, through a product or service that damages the environment. It was introduced to limit the effect on the climate of the consumption of polluting goods and services, and thus fight against global warming by promoting energy savings and less polluting energies. The "polluter pays" principle remains the basis of Eco-taxation. This principle consists of pollutants contributing to limiting pollution and damage to the environment.

Environmental taxation occupies an important place in the tax policies of member countries of the European Union. This part provides an overview of environmental taxation in Europe in general, based on numerous taxes that are part of the list of taxes included in ecological taxation. Recourse to European experiences, in particular the case of France, constructs an evaluation of the efficiency and consistency of ecological taxation.

In recent years, environmental taxation has evolved to support the ecological transition. The government in France has put in place an arsenal of regulations with stakeholders to support these developments. We will present the main taxes made up of Eco taxation in France, and an analysis of the performance of the French experience in ecological taxation.

II-1 The main taxes constituted Eco-taxation in France:

Environmental taxation in France can be divided into four categories depending on the function of the tax [6].

1- **The taxes themselves,** which are compulsory levies without compensation and the basis of which is a polluting product. This is the case, for example, with the tax on polluting activities (TGAP), based on emissions to the air or pesticides, and the TIPP.

2- **Charges that cover costs** for environmental services, mainly in the areas of water and waste.

3- **So-called positive** measures such as tax credits which seek in particular to orient investment choices in a more favorable direction for the environment.

4-**Tax incentives** (exemptions, deductions, rate cuts) which also seek to orient behavior in favor of the environment.

Eco-taxation in France can be divided into four categories depending on the type of tax; 4 following categories:

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- Energy taxes;
- Taxes on transport;
- Pollution taxes;
- Resource taxes.

In this part we will present a list of all the environmental taxes in force in France, based on the Eurostat nomenclature. For each tax, its base is specified in 2016.

TABLE 1. The Main Environmental Taxes in France.

Tax	Base
Energy	
Internal tax on the consumption of energy products - TICPE	Petroleum products used as fuels or fuels
Carbon component (integrated into ICT rates)	Fossil energies whose combustion emits CO2
Contribution to the public electricity service (CSPE)	Prorate of the quantity of electricity consumed
Local taxes on electricity (Internal tax on final electricity consumption TICFE + Tax on final electricity consumption TCFE)volt-amperes	Quantity of electricity subscribed (TICFE if greater than or equal to 250 kilo, otherwise TCFE)
Flat-rate tax on network companies (IFER)	9 components: Wind turbines and tidal turbines, nuclear or thermal installation, photovoltaic or hydroelectric installation, electrical transformers, radio stations, gas installations, SNCF railway equipment, RATP railway equipment, and certain telephone switching equipment
Internal tax on the consumption of natural gas –maintenance. TICGN	Natural gas used as fuel
Fuel tax in the overseas departments	Petroleum products used as fuel
Tax for the professional committee of strategic petroleum stocks	Costs of constitution and conservation during an Andes strategic stocks
Contribution of low-voltage electrical energy distributors low-voltage electrical energy distributors	Receipts from
Annual flat-rate tax on pylons	Pylons supporting power lines whose voltage is at least equal to 200 kilovolts
General tax on polluting activities (TGAP) fuels	Release for consumption of fuels
TIC on coal, lignite and coke	Quantity of energy delivered expressed in kWh
Transport	
Tax on registration certificates (gray cards)	Tax power of the vehicle
Additional tax on motor vehicle insurance the	proportional contribution to insurance premiums on motor vehicles
Tax on company cars	Number of vehicles detained individuals or leased by, or held by employees of the company and the miles driven which are subject to reimbursement fees
Tax due by motorway concessionaires	Number of kilometers traveled by users
Civil aviation tax	Number of passengers and mass of freight and mail loaded in France
Tax on the purchase of the most polluting new private vehicles (penalty of purchase)	Payable on the most polluting passenger cars, when they are first registered in France
Solidarity contribution on plane tickets	The number of passengers on board, excluding passengers in transit
Axle tax	Truck with an authorized weight equal to or greater than 12 tones, registered in France or in a third country (outside the EU) that has not concluded a reciprocal exemption agreement with France
Hydraulic Tax	Hydraulic works and hydroelectric works
Territorial solidarity contribution	Turnover relating to operations carried out for passenger rail transport services and commercial services directly related to them
Tax on ski lifts	Gross revenue from the sale of transport tickets ski lifts in mountain areas
Tax on pleasure boats (annual francization and navigation rights)	Ownership of a displaced vessel
Tax due by public air and sea transport companies	The number of passengers boarding in the Corsica and Guadeloupe regions , Guyana, Martinique and Reunion
Fee owed by the railway undertakings for the regulatory authority for railway activities	Part of the infrastructure use charge paid to SNCF signal within the limit of 5 thousandths + € 0.10 / km traveled on other lines of the rail network
Tax on maritime transport	Number of passengers on

at tination of protected natural areas	board destined for protected natural areas (list fixed by decree)
Annual tax on the possession of polluting private vehicles (annual penalty)	Vehicles emitting more than 245gCO2 / km (240gCO2 / km for vehicles registered since 2012)
Purchase tax on the most polluting second-hand	Vehicles Used vehicles emitting more than 200gCO2 / km or having a fiscal power greater than 10 horsepower
Tax intended to finance the development of vocational training actions in road transport	Additional tax on the issuance of vehicle registration documents for goods transport and public passenger transport
Pollution / resources	
Water pollution	charges The domestic pollution charge is based on the quantity of drinking water consumed: € 0.3 / m3
	The fee for the modernization of wastewater collection networks is based on the quantities of water used and sent to the collection networks: 0.5 € / m3
	The diffuse pollution charge concerns phytosanitary products (phytopharmaceuticals) and takes into account the toxicity, the dangerousness for the environment of the substances they contain: between 2 and 5 , 1 € / kg
	The charge for water pollution of non-domestic origin is based on the annual pollution discharged into the natural environment and relates to 10 constituent elements of pollution
	The charge for water pollution by livestock is based on the Large Cattle Unit (UGB) and takes into account the stocking rate (number of animals per hectare), by promoting extensive breeding: 3 € / UGB
General tax on polluting activities (TGAP) (excluding fuel TGAP)	TGAP waste: installations for the elimination of household and similar waste (incineration or storage) and special industrial waste: variation in rates depending on the environmental performance of the facilities (between € 4 / ton and € 150 / ton) TGAP emission pollutants in the atmosphere of certain substances: between € 5 / t (benzene) and € 1,000 / t (mercury) TGAP installations classified for environmental protection: between € 501.61 and € 2,525.35 per year and per TGAP installation lubricants oils and lubricating preparations: 44.02 € / t TGAP detergents: between 39.51 and 283.65 € / t TGAP extraction materials: 0.20 € / t
Water withdrawal	fees Annual volume of water withdrawal , expressed in m2
Municipal and departmental mining	Quantities of products extracted from mines, mines or quarries
Fees on other water uses	The fee for the protection of aquatic environments is based on fishing cards: € 10 per adult who engages in fishing, for one year
	The fee for water storage during low flow: the base is the volume of water stored during low flow; the ceiling rate of 0.01 € / m3 stored. The fee for obstacles on watercourses is due on structures constituting an obstacle on watercourses, blocking sediment transit and fish migration; the rate is set by the water agency for a maximum of 150 € / m of height difference.
Royalty due by the operators of liquid hydrocarbon mines	Value of the production of liquid or gaseous hydrocarbons at the start of the field (does not apply to deposits at sea)
Tax on sea	products Fishery products landed in France

Is environmental taxation a tool for protecting the environment? Gilles Rotillon In Different perspectives on the economy 2007/1 (n ° 1), page 109.

Source : CGDD, d'après les annexes au PLF 2017, Tome 1 de l'Évaluation des voies et moyens et rapport Agences de l'eau, et d'après le fichier Évaluation de recettes de la DGDDI.

II-2 the performance of the French experience in ecological taxation:

We can analyze the performance of Eco taxation in France by several forms and methods, in our work we choose to analyze it through their contribution to revenue budget, we will take as a reference the year 2016 data available for analysis.

TABLE 2. Total from environmental taxes, by tax category.

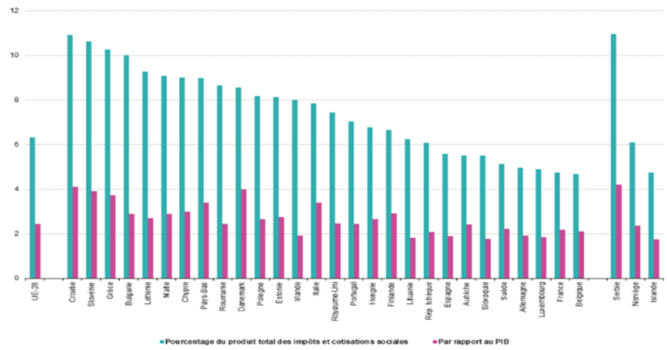
	In millions euros	in% of Total environ mental taxes	As a% of Gross Domestic Product	as a% of the total taxes and social contributi ons
Total des Taxes environnementales	359 294	100,00	2,4	6,3
Taxes sur l'énergie	275 392	76,6	1,9	4,8
Taxes sur les transports	71 269	19,8	0,5	1,3
Taxes sur la pollution / les ressources	12 633	3,5	0,1	0,2

EU-28, in 2015

Source: Eurostat (envactax) Ecotaxes

The environmental taxes in the European Union In 2015, the total public revenues arising from environmental taxes in the EU- 28 was 359.3 billion euros, or 2.4% of gross domestic product (GDP) and 6.3% of total government revenue from compulsory levies.

Fig. 1. Total product of environmental taxes, 2015 (in%)



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Source: Eurostat (envactax)

The product of environmental taxes for 2015 compared to GDP and to the total product of all taxes and social contributions by country. Relative to GDP, in 2015 the revenue from environmental taxes in the European Union reached the highest value in Croatia (4.1%), followed by Denmark with a ratio of 4.0%, Slovenia (3 , 9%) and Greece (3.7%). The lowest ratios between the product of environmental taxes and GDP (below 2%) were recorded in six Member States (Slovakia, Lithuania, Luxembourg, Spain, Ireland and Germany)

At the end of 2016 the highest tax revenue / GDP ratio in Europe was that of France, Belgium and Denmark The tax revenue / GDP ratio in France (48.4%), Belgium (47.3%) and Denmark (46.5%).[7] At the level of the OECD, the ranking of France is slightly better (among the 20th out of 31 member countries for which data is available). In France the internal tax on the consumption of energy products constitutes the fifth tax revenue, behind the value added tax (VAT), income tax, corporation tax and property tax.

TABLE 3 : The main environmental taxes in 2016

Name of the tax	**Revenue 2016 (in millions of euros)**	**Eurostat classification**
Internal tax on the consumption of energy products (TICPE)	28 456	Energy
Contribution to public service electricity (CSPE)	8264	
Local taxes on electricity	1588	
Flat-rate tax on network companies (IFER)	1592	
Domestic consumption tax on natural gas (TICGN)	1104	
Other energy taxes	1 310	
Tax on registration certificates (gray cards)	2 187	Transport
Additional tax on automobile insurance	996	
Tax due by motorway concessionaires	599	
Tax on company vehicles	542	
Civil aviation	410	
Other taxes on transport charges	1167	
Water pollution	1960	Pollution
General tax on polluting activities (TGAP) (waste, atmospheric pollution, etc.) excluding fuel TGAP	654	
Water levy fees	385	Resource
Other resource taxes	22	
Total (Eurostat field)	51 235	

Source: Volume I of ways and means of the 2018 finance bill, data from the General Directorate of Customs and indirect duties.

Source: Eurostat (envactax), press release in November 2018

According to these data, we see that environmental taxes (in the sense of Eurostat) represent 51 billion euros in 2016, we see that ecological taxation constitutes a transfer tax, where the revenue collected under an environmental tax will be allocated to the financing of an environmental public policy. We take, for example, the case of fees collected by water agencies, which are allocated to policies for managing water resources and improving their ecological and sanitary condition. So we will conclude that several environmental taxes have proven their effectiveness through their contribution to the financing of several public policies and to return a capital which makes it possible to revive activity and to make the economic system more efficient overall.

Axis III: Eco-Taxation in Morocco

Morocco has drawn up a negative balance sheet in terms of the environment, which calls for the economic priorities of the Moroccan State, to face this problem through economic and political instruments, and have taken several forms, notably legal, financial or fiscal, and depending on the nature of the environmental dimension (water, air, soil, waste, etc.). Among these main forms that promote environmental protection, environmental taxation. This part will be an essay of an analysis of environmental taxation in the Moroccan tax system. It wonders about the state of play of environmental taxation in Morocco? And to know is the Moroccan environmental taxation a viable solution to fight against climate change?

The concept of environmental taxation:

The concept of environmental taxation was first mentioned in Morocco in 2014 in a publication in the official bulletin N ° 6240 of 20-06-2014 of du Dahir n ° 1-14-09 of 4 joumada l in 1435 (March 6, 2014) promulgating the framework law n ° 99-12 on the national charter of the environment and sustainable development, that the concept of environmental taxation, and this aforementioned by article 30 "Is instituted an environmental taxation system composed of ecological taxes and fees imposed on activities characterized by a high level of pollution and consumption of natural resources. These taxes and fees can be applied to any characterized behavior, individual or collective, harming the environment and infringing the principles and rules of sustainable development”.

The state of play of environmental taxation in Morocco:

The industrial evolution of Morocco generated by the massive exploitation of the means of production having negative side effects, and contributes to the acceleration of the degradation of the environment. Statistics show this negative balance of the cost of environmental degradation in Morocco, in particular in 2000, the World Bank conducted a study entitled “Assessment of the cost of environmental degradation in Morocco” (CDE). This study evaluated, for the first time, the cost of environmental degradation which was estimated, for the year 2000, at 3.7% of GDP.[8]in 2014 The cost of environmental degradation for Moroccan society was estimated at nearly 32.5 billion dirhams, or 3.52% of GDP.

TABLE 4 : The damage caused by the cost of environmental degradation in Morocco.

	Lower bound	Upper Terminal Billions of dh	Average value	Average value % Gross Domestic Product
Water	11.10	12.20	11.70	1.26%
Air	6.30	13.10	9.70	1.05%
Soils	4.60	5.30	5.00	0.54%
Waste (including hazardous waste)	3.70	3.70	3.70	0.40%
Littoral	2.50	2.50	2.50	0.27%
Drills	0.00	0.00	0.00	0.00%
costs for Moroccan society	28.30	36.80	32.50	3.52%
Carbon emissions	4.60	25.40	15.00	1.62%
Cost for the global environment	4.60	25.40	15.00	1.62%

Source: World Bank

World Bank Report, The Cost of Environmental Degradation in Morocco Lelia Croitoru and Maria Sarraf (editors), January 2017.

Among the national costs, water pollution (1.26% of GDP) is the main vector of water degradation, followed by air pollution (1.05% of GDP. Land degradation also entails considerable costs (0.54% of GDP), in particular due to erosion of cultivated land, land clearing and the desertification of rangelands. Waste represents a relatively high cost (0.4% of GDP).This critical situation shown by these studies to assess the cost of environmental degradation, in particular the studies based on carried out by the World Bank in 2006 and 2017, urges the Moroccan State to establish an effective environmental policy.

Moroccan regulations on green taxation:

The Moroccan tax system is made up of an arsenal of taxes and duties having an impact on the environment. n the effects of the environment and exemptions and tax incentives encouraging the protection of the environment, we will include the main laws and codes:

- The General Tax Code (CGI) provides for measures to be exempt from the Annual Motor Vehicle Tax (TSAVA); vehicles intended for the public transport of person’s whose total laden weight or the maximum total towed laden weight is less than or equal to 3,000 kilograms and electric motor vehicles and hybrid motor vehicles (electric and thermal); the subjection of sales of solar water heaters to value added tax (VAT) at the reduced rate of 10%.
- The Code of Customs and Indirect Taxes (CDII) requires the royalty on the exploitation of phosphates which was abolished from January 1, 2008 (article 7 of the finance law for the year 2008); internal taxes on energy products; the ecological tax on plastics; the special cement tax; the Special Tax on Concrete Iron; the Special Tax on Sand.

• Law 47-06 relating to the taxation of local authorities constituted by the tax on the extraction of quarry products; the parking fee; driver's license tax; tax on taxi and coach licenses; the tax for checking vehicles over 5 years old; the tax on motorcycles with a cylinder capacity equal to or greater than 125 cm3. Established the rental value as the basis for calculating the business tax is capped at 50 million Dhs (this ceiling was 100 million Dhs from July 1, 1998 to 2001).

These measures are intended to limit large polluting industrial companies. To confirm the objective of national strategies against pollution and the effects of climate change. "Green, national and territorial taxation to help make Morocco a regional green factory "Morocco has adopted an ambitious policy of environmental protection and clean energy production. This policy is certainly a source of opportunities, both for developing the country's attractiveness and for generating new activities. Anticipation of this problem when planning activity zones, by providing them with the most advanced environmental characteristics, would make it possible to benefit from a rationalization of costs through the scale effect and at the same time from reduce investment procedures and improve Morocco's attractiveness for the launch of new economic projects, as part of an integrated vision of 'Morocco Regional Green Factory', Taxation, with its two dimensions, national and local , should play a direct role, in support of this ambition, which is likely to create activities with high added value and quality jobs " [9],.

The Main Environmental Taxes in Morocco:

The Moroccan tax system made up of a number of taxes that are favorable to the environment, we will mainly present:

- Taxes and taxes in favor of the environment
- The tax for checking vehicles over 5 years old;
- The tax on motorcycles with certain engine capacity;
- Internal taxes on energy products;
- Taxes on the extraction of quarry products;
- The wastewater treatment charge;
- The tipping fee;
- The special cement tax;
- The ecological tax on plastics;
- The Special Tax on Concrete Iron;
- The tax on the deterioration of pavements;
- The Special Tax on Sand, ...

Main tax exemptions and reductions favorable to the environment:

- Reduction of VAT on the rental of water and electricity meters;
- The reduction of VAT on the economy car; and solar water heaters
- The suspension of import VAT on butane gas; Etc**.**

It has been observed that the measures of environmental taxation either in the institutional or legal plan, are new adopted or require an effort to further explain these measures, but it should be noted that Morocco has taken a big step to establish taxation with the objective of combating environmental degradation and promoting sustainable development.

General conclusion:

To conclude this work, we can notice that environmental taxation is a tool and instrument to deal with climate change, but remains a difficult procedure to implement like all tax regulations, green taxation was introduced in the payroll of the European Union since the 1970s, this aspect has been developed either at the level of the differentiation of taxes and levies, and to broaden the tax revenue, or at the level of the expenditure target by the financing of actions environmental protection.

Morocco is following a journey to implement environmental taxation and then generalize it in the Moroccan tax system. We note that Morocco has taken a big step forward in establishing procedures and laws for the establishment of taxes related to environmental protection, such as framework law n ° 99-12 on the National Charter Environment and Sustainable Development encourages, but the real debate which aims to analyze the revenues of environmental taxation, either at the level of fundraising, or at the level of targeted expenditure.

Moroccan companies accept the introduction of Eco taxes but they ask that the funds collected should be paid into actions for the protection of the environment and not to be paid into the state budget. At this level, we really have to question the efficiency of the management of these funds.

REFERENCES

[1] The Organization for Economic Cooperation and Development. (L'Organisation de Coopération et de Développement Economiques).

[2] Taxation And Environment Twenty-Third Report To The President Of The Republic, 2005, Page 13.

[3] The General Tax On Polluting Activities . (la taxe générale sur les activités polluantes.)

[4] https://www.boursorama.com/patrimoine/actualites/l-ecofiscalite-qu-est-ce-que-c-est c46ffc1e77e863445a8658a10c9ac0 Consulté le 23-07-2020 à12h.

[5] arthur cecil pigou,m.a, the macmillan and co., limited st. martin's streety, london,1920.

[6] Is environmental taxation a tool for protecting the environment? Gilles Rotillon In Perspectives on the economy 2007/1 (n ° 1), page 109.

[7] Eurostat (Envactax), Press Release In November 2018

[8] The World Bank's Report, The Cost Of Environmental Degradation In Morocco Lelia Croitoru And Maria Sarraf (Editors), January 2017.

[9] Report of the Economic, Social and Environmental Council, "A Fiscal System, pillar for the New Development Model" prepared by the Permanent Commission in charge of Economic Affairs and Strategic Projects Page 103

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