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FOREWORD

The International Journal on Optimization and Applications (IJOA) is an open access, double blind peer-reviewed online journal aiming at publishing high-quality research in all areas of : Applied mathematics, Engineering science, Artificial intelligence, Numerical Methods, Embedded Systems, Electric, Electronic engineering, Telecommunication Engineering... the IJOA begins its publication from 2021. This journal is enriched by very important special manuscripts that deal with problems using the latest methods of optimization. It aims to develop new ideas and collaborations, to be aware of the latest search trends in the optimization techniques and their applications in the various fields..

Finally, I would like to thank all participants who have contributed to the achievement of this journal and in particular the authors who have greatly enriched it with their performing articles.

Prof. Dr. Hanaa HACHIMI
Editor in chief
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On the existence result of fuzzy fractional boundary value problems

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***Abstract*—In this paper we investigate the existence result of solutions for boundary value problem of nonlinear fuzzy fractional differential equations involving Caputo fuzzy fractional derivatives. We conclude our work by presenting an illustrative example.**

***Index Terms*—Fuzzy numbers,Fuzzy functions,fuzzy fractional integral,fuzzy fractional caputo derivative.**

I. Introduction

Fractional differential equations (DEs) have received considerable attention in the recent years due to their wide applications in the areas of applied mathematics, physics, engineering, economy, and other fields. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional DE [6], [7], [8], [9], [10], [11].In general, most of fractional DEs do not have exact solutions. Particularly, there is no known method for solving fractional boundary value problems (BVPs) exactly. As a result, numerical and analytical techniques have been used to study such problems. It should be noted that much of the work published to date concerning exact and numerical solutions is devoted to the initial value problems for fractional order ordinary DEs. The theory of BVPs for fractional DEs has received attention quiet recently. The attention drawn to the theory of existence and uniqueness of solutions to BVPs for fractional order DEs is evident from the increased number of recent publications.In the book by Kelley and Peterson [1] the following result is established:

Theorem 1: ([1],Theorem 7.7).Assume $f : [a;b]\times R \longrightarrow R$ is continuous and satisfies a uniform Lipschitz condition with respect to the second variable on $[a;b]\times R$ with Lipschitz constant K; that is,

$$\mid f(t,x)-f(t,y)\mid \leq K \mid x-y\mid$$

for all $(t,x),(t,y)\in[a;b]\times R$.if

$$b-a<\frac{2\sqrt{2}}{\sqrt{K}}$$

then the boundary valued problem

$$y''(t)=f(t,y(t)),\quad t\in[a,b]$$

$$y(a)=A, y(b)=B\quad A,B\in R$$

has a unique continuous solution.

In this work we want to extend the above result by considering a fractional Riemmann-Liouville derivative (we refer the reader to [5] for the defnitions and basic results on fractional calculus) instead of the classical operator y'',i.e., we prove the existence and uniqueness of solutions for the fuzzy fractional diferential boundary value problem.

$$D^{\alpha}x(t)=f(t,x(t))\quad t\in[a,b] \tag{I.1}$$

$$x(a)=\tilde{0}, x(b)=B\quad B\in E^1 \tag{I.2}$$

Where $1<\alpha\leq 2$ and E^1 is the collection of fuzzy numbers.

II. Preliminaries

Definition 1: [4] A fuzzy number is mapping $u:\mathbb{R}\longrightarrow[0,1]$ such that

1) u is upper semi-continuous
2) u is normal,that is, there exist $x_0\in\mathbb{R}$ such that $u(x_0)=1$
3) u is fuzzy convex, that is, $u(\lambda x+(1-\lambda)y)\geq \underline{\min\{u(x),u(y)\}}$ for all $x,y\in\mathbb{R}$ and $\lambda\in[0,1]$.
4) $\overline{\{x\in\mathbb{R},\ u(x)>0\}}$ is compact.

The $\alpha-Cut$ of a fuzzy number u is defined as follows:

$$[u]^{\alpha}=\{x\in\mathbb{R}/u(x)\geq\alpha\}$$

We denote by E^1 the collection of all fuzzy numbers.

Definition 2: [4] A fuzzy number u in a parametric form is a pair of function $(\underline{u}(r),\overline{u}(r))$ with $r\in[0,1]$,which satisfy the following requirements:

1) $\underline{u}(r)$ is a bounded nonincreasing lef continuous function in $[0,1]$.
2) $\overline{u}(r)$ is a bounded nondecreasing lef continuous function in $[0,1]$.
3) $\underline{u}(r)\leq\overline{u}(r)\ \forall r\in[0,1]$

Moreover,we also can present the $r-cut$ representation of fuzzy number as $[u]^r=[\underline{u}(r),\overline{u}(r)]$.

Definition 3: Let $x,y\in E^1$,if there exists $z\in E^1$ such that,$x=y+z$ then z is called the Hukuhara difference of x and y, denoted by $x\ominus y$.

Definition 4: According to the Zadeh's extension principle, the addition on E^1 is defined by

$$(u\oplus v)(z):=\sup_{z=x+y} min\{u(x),v(y)\}$$

And scalar multiplication of a fuzzy number is given by

$$(k\odot u)(x):=\begin{cases} u(x/k) & ,k>0\\ \widetilde{0} & ,k=0\end{cases}$$

Definition 5: [4] Let $f : [a,b] \to E^1$ and $t_0 \in [a,b]$.We say that f is Hukuhara differentiable at t_0 if there exists $f'(t_0) \in E^1$ such that:
(1) $f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0+h)\ominus f(t_0)}{h}$
$= \lim_{h\to 0^-} \frac{f(t_0)\ominus f(t_0-h)}{h}$
Or
(2) $f'(t_0) = \lim_{h\to 0^+} \frac{f(t_0)\ominus f(t_0+h)}{-h}$
$= \lim_{h\to 0^-} \frac{f(t_0-h)\ominus f(t_0)}{-h}$

Proposition 1: Let $f : [a,b] \longrightarrow E^1$ be a function such that $[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0,1]$

1) If f is (1)-differentiable function,then $[f'(x)]^r = \left[\underline{f}'(x;r), \overline{f}'(x;r)\right]$
2) If f is (2)-differentiable function,then $[f'(x)]^r = \left[\overline{f}'(x;r), \underline{f}'(x;r)\right]$

Definition 6: [4] Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1$ with $r \in [0,1]$,then the Hausdorf distance between u and $v$is given by
$D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$

Proposition 2: [4] D is a metric on E^1 and has the following properties:

1) $(E^1; D)$ is a complete metric space.
2) $D(u+w, v+w) = D(u,v)$, $\forall u,v,w \in E^1$.
3) $D(ku,kv) = |k|D(u,v)$, $\forall u,v \in E^1$ and $k \in \mathbb{R}$.
4) $D(u+w, v+z) \le D(u,v) + D(w,z)$, $\forall u,v,w,z \in E^1$.

We denote by $\mathcal{C}^F = \mathcal{C}([a,b], E^1)$ space of all fuzzy-valued functions which are continuous on $[a,b]$,and $\mathcal{P}_K(\mathbb{R})$ is the collection of all the compact subset of $\mathbb{R}$.

Definition 7: $F : [a,b] \longrightarrow E^1$ is strongly measurable if $\forall \alpha \in [0,1]$,the set-valued mapping $F_\alpha : [a,b] \longrightarrow \mathcal{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable A function $F : [a,b] \longrightarrow E^1$ is called integrably bounded, if there exists an integrable function h such that $|x| < h(t)$ $\forall x \in F_0(t)$.

Definition 8: Let $F : [a,b] \longrightarrow E^1$. The integral of F on $[a,b]$ denoted by $\displaystyle\int_a^b F(t)dt$,is given by
$\left[\int_a^b F(t)dt\right]^\alpha =$
$\left\{\int_a^b f(t)dt \mid f : [a,b] \longrightarrow \mathbb{R}\text{is a measurable selection for} F_\alpha\right\}$
for all $\alpha \in [0,1]$.

Definition 9: [5] Let $f : [a,b] \longrightarrow E^1$ and $0 < \alpha < 1$,the fuzzy Riemann-Liouville fractional integral is defined by

$$I^\alpha_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$$

Remark 1: Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0,1]$,then
$\left[I^\alpha_{a+} f(t)\right]^r = \left[I^\alpha_{a+}\underline{f}(x;r), I^\alpha_{a+}\overline{f}(x;r)\right]$
Where $I^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\underline{f}(s;r)}{(t-s)^{1-\alpha}} ds$
And
$I^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\overline{f}(s;r)}{(t-s)^{1-\alpha}} ds$

Definition 10: [5] Let $f : [a,b] \longrightarrow E^1$,$x_0 \in [a,b]$ and $\phi(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x \frac{f(s)}{(t-s)^{1-\alpha}} ds.$
The function f is called fuzzy Riemann-Liouville fractional differentiabl of order $0 < \alpha < 1$ at x_0 if there exists an element $D^\alpha_a f(x_0) \in E^1$ such that
(1) $D^\alpha_a f(x_0) = \lim_{h\to 0^+} \frac{\phi(x_0+h)\ominus\phi(x_0)}{h}$
$= \lim_{h\to 0^-} \frac{\phi(x_0)\ominus\phi(x_0-h)}{h}$
Or
(2) $D^\alpha_a f(x_0) = \lim_{h\to 0^+} \frac{\phi(x_0)\ominus\phi(x_0+h)}{-h}$
$= \lim_{h\to 0^-} \frac{\phi(x_0-h)\ominus\phi(x_0)}{-h}$
For the sake of simplicity, a fuzzy-valued function f is $^{RL}[(1)-\alpha] - differentiable$ if it is differentiable,as in definition(2.6), Case (1), and is $^{RL}[(2)-\alpha] - differentiable$ if it is differentiable as in definition (2.6),Case (2).

Remark 2: [5] Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0,1]$,then we have the following relations:
1)If f is $^{RL}[(1)-\alpha] - differentiable$fuzzy valued function then,
$\left[D^\alpha_{a+} f(t)\right]^r = \left[D^\alpha_{a+}\underline{f}(x;r), D^\alpha_{a+}\overline{f}(x;r)\right]$
1)If f is $^{RL}[(2)-\alpha] - differentiable$fuzzy valued function then,
$\left[D^\alpha_{a+} f(t)\right]^r = \left[D^\alpha_{a+}\overline{f}(x;r), D^\alpha_{a+}\underline{f}(x;r)\right]$
Where $D^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t \frac{\underline{f}(s;r)}{(t-s)^{1-\alpha}} ds$
And
$D^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t \frac{\overline{f}(s;r)}{(t-s)^{1-\alpha}} ds$

Definition 11: [5] Let $f : [a,b] \longrightarrow E^1$ and $x_0 \in [a,b]$.
The function f is called fuzzy Caputo fractional differentiable of order $0 < \alpha < 1$ at x_0 if there exists an element $^cD^\alpha_a f(x_0) \in E^1$ such that:

$$^cD^\alpha_a f(x_0) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{1-\alpha}} ds \qquad \text{(II.1)}$$

Then we say f is $^c[(1)-\alpha] - differentiable$ if f is (1)-differentiable, and f is $^c[(2)-\alpha] - differentiable$ if f is (2)-differentiable.

Remark 3: [5] Since$[f(x)]^r = \left[\underline{f}(x;r), \overline{f}(x;r)\right]$ for each $r \in [0,1]$,then
1) If f is (1)-differentiable then,
$\left[^cD^\alpha_{a+} f(t)\right]^r = \left[^cD^\alpha_{a+}\underline{f}(x;r), ^cD^\alpha_{a+}\overline{f}(x;r)\right]$
2)If f is (2)-differentiable then,
$\left[^cD^\alpha_{a+} f(t)\right]^r = \left[^cD^\alpha_{a+}\overline{f}(x;r), ^cD^\alpha_{a+}\overline{f}(x;r)\right]$
Where $^cD^\alpha_{a+}\underline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\underline{f}'(s;r)}{(t-s)^{1-\alpha}} ds$
And
$^cD^\alpha_{a+}\overline{f}(t;r) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\overline{f}'(s;r)}{(t-s)^{1-\alpha}} ds$

III. Main Results

We start by writing the boundary value problem (I.1)(I.2) in its integral form.

Lemma 1: Suppose that f is a continuous function. A function $x \in C^F([a;b])$ is a solution of $(I.1)$ $(I.2)$ if and only if x satisfes the integral equation

$$x(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds \qquad \text{(III.1)}$$

$$G(t,s) = \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1} & a \le s \le t \le b \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} & a \le t \le s \le b \end{cases}$$

Proof 1:

Proof 2: By using the parametric form of fuzzy number we have $x(t) = (\underline{x}(r), \overline{x}(r))$,then the problem (1.1) ,(1.2) is equivalent to

$$(1.3) \begin{cases} D^{\alpha}\underline{x}(t;r) = \underline{f}(t, \underline{x}(t;r);r) & t \in [a,b] \\ \underline{x}(a;r) = \underline{0}(r), \quad \underline{x}(b;r) = \underline{B}(r) \end{cases}$$

$$(1.4) \begin{cases} D^{\alpha}\overline{x}(t;r) = \overline{f}(t, \overline{x}(t;r);r) & t \in [a,b] \\ \overline{x}(a;r) = \overline{0}(r), \quad \overline{x}(b;r) = \overline{B}(r) \end{cases}$$

It is well known that solving $(I.3)$ is equivalent to solving the integral equation
$\underline{x}(t;r) = c\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + d\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-2)} + \int_a^t G(t,s)\underline{f}(t, \underline{x}(t;r);r)ds$
where c and d are some real constants.Now,$d = 0$ by the first boundary condition.On the other hand, $\underline{x}(b;r) = \underline{B}(r)$ implies.
$\underline{B}(r) = c\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \int_a^b (b-s)^{\alpha-1}\underline{f}(s, \underline{x}(s;r);r)ds$
which after some manipulations yields
$c = \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\underline{B}(r) - \int_a^b (b-s)^{\alpha-1}\underline{f}(s, \underline{x}(s;r);r)ds\right)$.
$\underline{x}(t;r) = \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left(\underline{B}(r) - \int_a^b (b-s)^{\alpha-1}\underline{f}(s, \underline{x}(s;r);r)ds\right)$
$\times\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} - \int_a^t (t-s)^{\alpha-1}\underline{f}(t, \underline{x}(t;r);r)ds$
and the proof is complete.

Proposition 3: Let G be the Green function given in Lemma (1) Then

$$\int_a^b \mid G(t,s) \mid ds \le \frac{(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1}$$

Proof 3: It is known [2]Lemma (2.2) that $G(t,s) \ge 0$ for all $t, s \in [a,b]$ Therefore
$\int_a^b \mid G(t,s) \mid ds =$
$\frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}\right)ds$
$+\frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}\right)ds$
$= \frac{1}{\Gamma(\alpha)}\left(-\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha-1}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha}}\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-a)^{\alpha-1}}{\alpha}\right)$
$+\frac{1}{\Gamma(\alpha)}\left(-\frac{(t-a)^{\alpha-1}}{\alpha} + \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\frac{(b-t)^{\alpha-1}}{\alpha}\right)$
$= \frac{1}{\Gamma(\alpha)}\frac{(t-a)^{\alpha-1}(b-t)}{\alpha}$
We define $g : [a,b] \to R$ by

$$g(t) = \frac{(t-a)^{\alpha-1}(b-t)}{\alpha}$$

Diferentiating the function g we immediately find that its maximum is achieved at the point

$$t^* = \frac{(\alpha-1)b+a}{\alpha}$$

Moreover

$$g(t^*) = \frac{(a-1)^{\alpha-1}(b-a)^{\alpha}}{\alpha}$$

wich complete the proof

Theorem 2: Assume$f : [a;b] \times E^1 \longrightarrow R$ is continuous and satisfes a uniform Lipschitz condition with respect to the second variable on $[a,b] \times E^1$ with Lipschitz constant K that is,

$$D\left(f(t,x(t)), f(t,y(t)\right) \le KD(x,y)$$

For all $(t,x),(t,y) \in [a;b] \times E^1$.
If $\frac{K(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1} < 1$ then the boundary-value problem

$$D^{\alpha}x(t) = f(t,x(t)) \quad t \in [a,b]$$

$$x(a) = \tilde{0}, x(b) = B \quad B \in E^1$$

has a unique continuous solution.

Proof 4: Let C^F the complete metric space of fuzzy continuous functions defined on $[a,b]$ with the distance D By Lemma (2.1),$y \in C^F$ is a solution of $(II.2)$,$(II.3)$ if and only if it is a solution of the integral equation

$$y(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds$$

Let $T : C^F \longrightarrow C^F$ defined by

$$Ty(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}B + \int_a^b G(t,s)f(t,y(s))ds$$

for $t \in [a;b]$.We will show that the operator T has a unique fixed point.
Let $x, y \in C^F$,then

$$\begin{aligned} D(Tx(t),Ty(t)) &\le \int_a^b |G(s,t)|D(f(x(s),s), f(y(s),s)))ds \\ &\le \int_a^b |G(s,t)|KD(x(s),y(s))ds \\ &\le \frac{K(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(b-a)^{\alpha-1}D(x,y) \end{aligned}$$

where we have used Proposition 3 By (2.1) we conclude that T is a contracting mapping on C_F and by the Banach contraction mapping theorem we get the desired result.

Remark 4: We note that when $\alpha = 2$ in Theorem 2, one immediately obtains Theorem 1 (apart from the restriction $A = 0$ $(y(a) = 0)$, which we have to assume in order to consider continuous solutions on $[a,b]$ to $(II.2)$.

Example 1: As an example we consider the initial-value problem

$$D^{3/2}(\underline{x}(t;r), \overline{x}(t;r)) = (sin(\underline{x}(t;r)), sin(\overline{x}(t;r))) \quad t \in [0,1] \qquad \text{(III.2)}$$

$$(\underline{x}(0;r), \overline{x}(0;r)) = (0,0) \quad (\underline{x}(1;r), \overline{x}(1;r)) = 0. \qquad \text{(III.3)}$$

Here $f(t, x(t;r);r) = sin(\underline{x}(t;r))$.
And $|sin(x(t))| \leq 1 = K$.
Since $\alpha = 3/2$ we have.

$$\frac{1(\alpha-1)^{\alpha-1}}{\Gamma(\alpha)\alpha^{\alpha+1}}(1-0)^{\alpha-1} = \frac{3}{4}\pi^{\frac{1}{3}}3^{2/3}$$

The condition of theorem 2 is satisfied,thus the initial-value problem $(III.2)$ $(III.3)$ has a unique solution.

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Acceptance of public procurement digitalization using the TAM Technology Acceptance Model (Davis, 1989)

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Abstract—

Digitalization plays a key role in society, it is considered as an important lever to ensure financial inclusion, currently, digitalization has become a must for a company or any type of organization. Indeed, it applies to all areas and ensures an optimization of time and money. In Morocco we note the existence of several projects in the digital with the support of several international organizations like the World Bank

The purpose of this paper is to examine the acceptance of digitalization in public procurement using the TAM (Davis, 1989) acceptance model in a Moroccan context. Using the questionnaire developed by Wong and Teo (2009), we conducted a survey of 200 Moroccan companies working in the field of cleaning and security. The results showed that the TAM can be considered as a valid instrument to study the factors that define and influence the acceptance of the digitalization by the Moroccan cleaning and security companies. They also show that the dimensions of the TAM are articulated in the same way as Davis (1989) and Wong and Teo (2009) have recommended.

Keywords— TAM model, Digitalization, Public procurement, perception.

I. Introduction

Currently, digitalization in the field of public procurement plays an essential role in establishing transparency. Morocco has opted for the implementation of a system of digitalization of public procurement, and this is considered an instrument of good governance of public finances.

Indeed, the State aims to simplify, improve the performance of procurement, efficiency and transparency of public procurement as positive outcomes of the digital transformation. As a result, the digitalization of public procurement minimizes the risk of mistakes, controls the time required to collect and process data, and makes the management process more secure and transparent.

The digitalization of public procurement procedures is based on three pillars: the public procurement portal represents a database of public procurement, the electronic submission which is a platform for supplier profiles and last but not least the e-auction which is a form of electronic group purchases.

However, the success of this digitalization requires the acceptance and participation of all the companies. But this digital transformation will not be successful if it is not complemented by a real commitment from companies and suppliers, so it would be important to support the users by providing the necessary tools.

In light of these realities, it is important to understand the factors that can influence the acceptance of digitalization among users (Moroccan companies) by using the TAM (technology acceptance model) (Davis, 1989).

The purpose of our study is to test the effectiveness of the TAM as a model that might explain the intention to use digitalization by Moroccan cleaning and security companies.

II. Theoretical Context & Methodology

A. *Theoretical Context*

In the following section, we will review the different theories that have studied this aspect since the seventies of the last century until the latest productions on the topic, in order to mark out the sphere of the variables that can influence the new technologies user's behavior.

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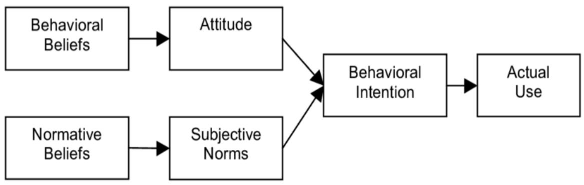
1. **Theory of Reasoned Action (Ajzen & Fishbein, 1980 ; Fishbein & Ajzen, 1975) [1] :**

This model presumes and assumes that all behavior is initiated by a behavioral intention, which means that the decisions are purely rational. There are two components of the behavioral intention. first, the attitude towards the behavior and second, the subjective norm.

Figure 1.Theory of Reasoned Action (Ajzen & Fishbein, 1980 ; Fishbein & Ajzen, 1975)



Source: Theory of Reasoned Action (Ajzen & Fishbein, 1980).

The attitude towards the behaviour would be determined by people's beliefs about the consequences of that behaviour multiplied by the evaluation of those consequences. The beliefs are defined by the individual's subjective probability that performing a particular behavior will generate some specific outcome.

The subjective norm translates the social pressure submitted by the individual in the process of the decision making (emission of the behavior); this pressure is concretized practically by the taking into account the manner in which the important people could react facing the emission of some behavior.

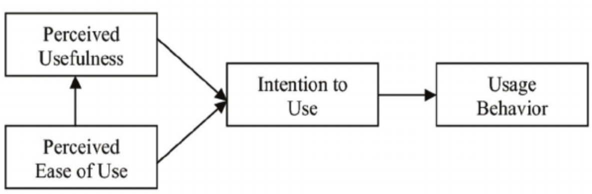
The theory of reasoned action assumes that all other factors that influence behavior are done so indirectly, such as by influencing attitudes toward the behavior or subjective norms.

However, the scope of the theory of reasoned action remains limited to behaviors under intentional control.

2) Technology Acceptance Model 1989 (TAM1) [2] :

The Technology Acceptance Model (TAM) proposed by DAVIS in 1989 has become the most dominant model of the acceptability of the technologies the information and the communication. It aims to explain the adoption or not of technology.

Figure 2.Technology Acceptance Model 1989 (TAM1)



Source: Fred D. Davis, Richard P. Bagozzi et Paul R. Warshaw, « User Acceptance of Computer Technology: A Comparison of Two Theoretical Models », 1989.

The TAM has two fundamental determinants of the acceptance of the technologies: The perceived usefulness and the perceived ease of use.Davis has defined perceived utility as: "The prospective user's subjective probability that using a specific application system will increase his or her job performance within an organizational context".

As for perceived ease of use, it was defined as follows: "The degree to which the prospective user expects the target system to be free of effort".

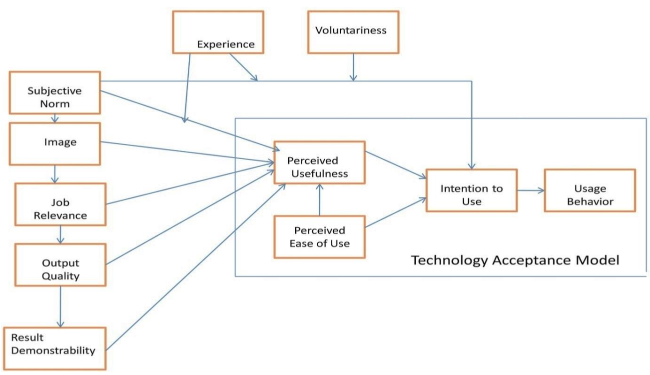
3) Technology Acceptance Model 2000 (TAM2) [3] :

In 2000, Viswanath Venkatesh and Fred D. Davis published the results of a study they conducted on four organizations to test the theoretical extension of the first version of Davis' 1989 technology acceptance model.

They have measured the TAM constructs three times; the first one before the implementation of the information system, the second one after one month of its implementation and the last one three months after the implementation. These three measuring points allowed us to identify the power of the TAM in predicting the users' behavior for all the studied organizations.

The year 2000 model represents an upgrade to TAM 1 because it presents several components of the perceived usefulness (PU), which represents the major construct of the TAM in addition to the perceived ease of use, the second most important component of the intention to use in the basic TAM model.

Figure 3.Technology Acceptance Model 2000 (TAM2)



Source : Viswanath Venkatesh et Fred D. Davis, « A Theoretical Extension of the Technology Acceptance Model: Four Longitudinal Field Studies ».

In this model Davis and Venkatesh enumerated seven external elements of the TAM that have a direct influence on the perceived usefulness. They are divided into two sections:

first, the social influence process that includes the subjective norm, voluntarism and image.

Second, the instrumental cognition process that includes the relevance of the work, the quality of the output, the demonstrability of the results and the perceived ease of use.

4) Unified Theory of Acceptance and Use of Technology 2003 (UTAUT) (TAM3) [4] :

Developed by Venkatesh et al.(2003) it' s a synthesized and complete theory, which seeks to test and analyze eight pre-existing theories of technology acceptance, which are; ; TAM (Davis 1989), Diffusion of Innovations Theory (Rogers 1962), TAR (Ajzen and Fishbein, 1975), Motivation Model (Davis 1992), TCP (Ajzen 1991), MAT and TCP combined (Taylor and Todd 1995), PC Use Model (MPCU), and Social Cognitive Theory (Bandura 1989, Compeau and Higgins 1995).

By bringing together, consolidating and refining the previously established theories, UTAUT is considered by its authors to be a model that provides the best account of the adoption and use of technology.

UTAUT assumes that the actual use of a technology is a function of the intention to use, which is influenced by the following components: the expected performance, the expected effort, the social influence, and the facilitating conditions.

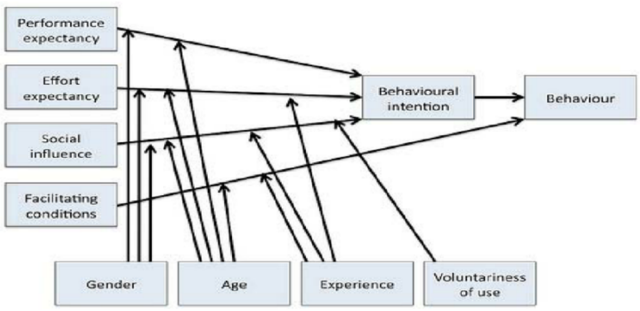
The expected performance is the gain a user expects to achieve by using the technology .

The expected effort refers to the amount of effort a user has to make in his daily use of the technology.

The Social influence is a belief that other people will support the user in his choice to use technology.

And finally we have the facilitating conditions that are defined by Venkatesh as follows: "the degree to which an individual believes that an organizational and technical infrastructure exists to support the use of the System.

Figure 4.Unified Theory of Acceptance and Use of Technology 2003 (UTAUT) (TAM3)



Source: Viswanath Venkatesh, Michael G. Morris, Gordon B. Davis et Fred D. Davis, « User Acceptance Of Information Technology: Toward A Unified View ».

The UTAUT incorporates new categories of variables known as moderators that can change the influence of the component variables on the intention to use.

They are: gender, age, experience of use, and mandatory or voluntary.

5) Technology Acceptance Model 2008 (TAM3) [5] :

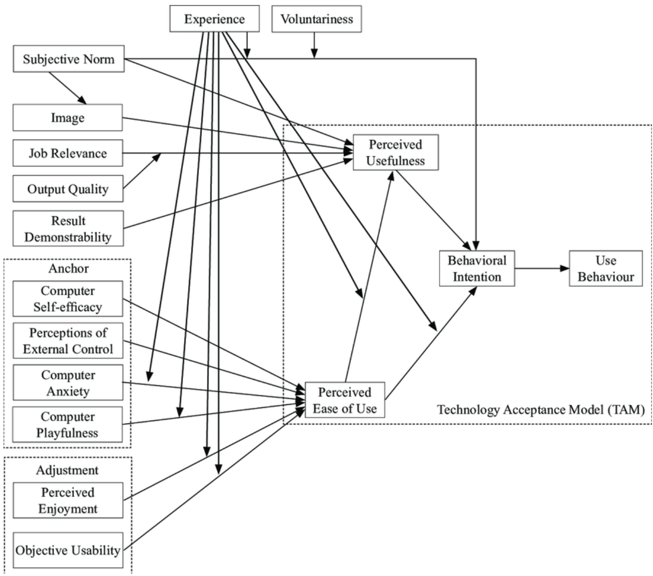
Despite the existence of a body of research that explains the factors that help ease the technology acceptance, many of the adoptions failed.

After much research Venkatesh was finally able to identify in 2008 one anomaly related to TAM that is considered to be the strongest technology acceptance model. This anomaly was clarified according to Venkatesh by the following question; "imagine talking to a manager and saying that to be adopted technology must be useful and easy to use. Imagine the reaction would be 'Duh!' The more important questions are what makes technology useful and easy to use".

TAM3 attempts to draw out all the variables that may influence the future use of any technology. Venkatesh and Bala combined the TAM2 (Venkatesh and Davis, 2000) with the perceived ease of use components (Venkatesh, 2000).

The goal of this production is to provide an integrative model of decision-making of a user with respect to the use of a technology by specifying all the components of perceived usefulness and perceived ease of use .

Figure 5.Technology Acceptance Model 2008 (TAM3)



Source : Greg Walker et Nancy Johnson, « Faculty intentions to use components for web-enhanced instruction ».

Venkatesh and Baia (2008) have combined the perceived usefulness components presented in TAM 2 with the perceived ease of use components suggested by Venkatesh (2000). they are distinguished into two categories :

1- Anchors: it represents the set of beliefs that one has about computers and/or their use. It includes four components: Computer self-efficacy, perceptions of external control, computer anxiety and computer playfulness.

2- Adjustment: it is the feedback of the users after using the system in question. It includes two components: Perceived enjoyment and objective usability.

The essence of this theoretical section is to synthesize all the different models related to technology acceptance in order to allow researchers to understand the mechanisms that may influence the behavior of the final user. Moreover, we want to help any organization wishing to introduce a new technology, to understand the acceptance issues it might be confronted to.

B. Research Hypotheses

Based on the TAM model and questionnaire developed by Wong and Teo (2009), we will test the following hypotheses:

H1: Perceived ease of use has a positive effect on the perceived usefulness of computers (digitalization) in Moroccan companies;

H2: Perceived ease of use has a positive effect on attitudes towards computers (digitalization) in Moroccan firms;

H3: Perceived usefulness has a positive effect on attitudes towards computers (digitalization) among Moroccan firms;

H4: Perceived usefulness has a positive effect on the intention to use computers (digitalization) among Moroccan firms;

H5: Attitudes towards computers (digitalization) has a positive effect on the intention to use computers (digitalization) among Moroccan firms.

C. Methodology

In order to test the mentioned hypotheses, we used a quantitative approach :

the study sample

The participants of this research are 200 Moroccan companies that provide cleaning and security services, to have a certain representativity, we realized the questionnaire on companies that are located all over the Moroccan territory.

Our questionnaire is made of 16 items, online format, the profiles of the respondents and their functions are listed in Table 1:

Table 1.Respondent information

Gender	Nbr	%
Male	68	34
Female	132	66
TOTAL	**200**	
Function	**Nbr**	**%**
Manager	52	26
Head of a department	82	41
Commercial	66	33
TOTAL	**200**	
Specialty	**Nbr**	**%**
Economics and Management	121	60.5
Right	52	26
Computer Science	22	11
ISTA Diploma	5	2.5
TOTAL	**200**	
Education level	**Nbr**	**%**
Baccalaureate or less	7	3.5
Bac +2	71	35.5
Bac +3	83	41.5
Bac +5	34	17

Source: Author's processing of survey data

Measuring Instrument:

In this research, we used Likert scale survey questions it's a 5-point scale developed by Wong and Teo (2009). The 16 items are affirmations with 5 answer options that range from "completely disagree" to "completely agree".

The questionnaire items are grouped into 4 axes as presented in Table II.

Table 2.List of dimensions and corresponding items

	Question	Code
Perceived usefulness (adapted from Davis, 1989)	The use of the public procurement portal will improve my work.	**A 1.1**
	Using the procurement portal will improve my efficiency.	**A 1.2**
	Using the procurement portal will increase my productivity.	**A 1.3**
	I find the procurement portal to be a useful tool in my work.	**A 1.4**
Perceived ease of use (adapted from Davis, 1989)	My interaction with the procurement portal is clear and understandable.	**B 1.1**
	I find it easy to manipulate the portal of the public markets I want.	**B 1.2**
	Interacting with the public procurement portal does not require much mental effort.	**B 1.3**
	I find the procurement portal easy to use.	**B 1.4**
Intention to use the computer (adapted from Davis, 1989)	I will be using the public procurement portal in the future.	**C 1.1**
	I make a plan to use the public procurement portal	**C 1.2**
	I will register for training to better use the public procurement portal	**C 1.3**
	I will seek support to master the use of the public procurement portal	**C 1.4**
Attitudes toward computer use.	The public procurement portal makes the job more interesting.	**D 1.1**
	Working with the public procurement portal is fun.	**D 1.2**
	I like working with the public procurement portal.	**D 1.3**
	I am looking for aspects of my job that require the use of the Public Procurement Portal.	**D 1.4**

Source: Source: Wong and Teo (2009).

Data analysis

The purpose of this work is to study the measurement instrument by using Cronbach's alpha score and Exploratory factor analysis (EFA) in order to evaluate the Dimensionality, Reliability and the Validity of the constructs.

The statistical analyses of the data were carried out under the software "Stata Corp Stata 14.2 (Revision 16 Nov 2016)".

Results

In this section we will present the results obtained by our research.

1. **The study of the measuring instrument**

In order to evaluate the reliability of our dimension, we will calculate the indicator of the reliability of the dimensions ALPHA of Cronbach, as shown in the following table we note that the values obtained are satisfying.

Table 3.Measures of Dimensional Fidelity

Dimension	Number of items	Cronbach's Alpha
Perception of usefulness	4	0.75
Perceived ease of use	4	0.72
Intention to use the computer	4	0.87
Attitudes towards computer use	4	0.79

Source: Author's processing of survey data

2. **Exploratory factor analysis**

In this step, we performed a factor analysis with the VARIMAX normalization Kaiser rotation on all the items retained during the reliability study.

The extraction method used is the principal axes factorization. Concerning the number of factors to be extracted.

Table 4.Exploratory factor analysis

		Factor			
Item	Code	1	2	3	4

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The use of the public procurement portal will improve my work.	A 1.1	,809			
Using the procurement portal will improve my efficiency.	A 1.2	,756			
Using the procurement portal will increase my productivity.	A 1.3	,903			
I find the procurement portal to be a useful tool in my work.	A 1.4	,590			
My interaction with the procurement portal is clear and understandable.	B 1.1				,790
I find it easy to manipulate the portal of the public markets I want.	B 1.2				,450
Interacting with the public procurement portal does not require much mental effort.	B 1.3				,868
I find the procurement portal easy to use.	B 1.4				,490
I will be using the public procurement portal in the future.	C 1.1			,811	
I make a plan to use the public procurement portal	C 1.2			,895	
I will register for training to better use the public procurement portal	C 1.3			,503	
I will seek support to master the use of the public procurement portal	C 1.4			,390	
The public procurement portal makes the job more interesting.	D 1.1		,689		
Working with the public procurement portal is fun.	D 1.2		,976		
I like working with the public procurement portal.	D 1.3		,358		
I am looking for aspects of my job that require the use of the Public Procurement Portal.	D 1.4		,607		

Source: Source: Wong and Teo (2009).

3. Tests of the research hypotheses

In this part we analyze by structural equations the causality links that exist between the dimensions of the research model.

Tested link	β	S.E	C.R	Validation of H
Perceived usefulness Perceived ease of use	,560	,113	5,120	H1 VALIDATED
Attitudes toward computer use Perceived ease of use	,648	,145	4,245	H2 VALIDATED
Attitudes toward computer use Perceived usefulness.	,501	,111	4,527	H3 VALIDATED
Intention to use the computer Attitudes toward using the computer	,367	,074	4,564	H5 VALIDATED
Intention to use the computer Perceived ease of use	,597	,112	5,186	H4 VALIDATED

Source: Author's processing of survey data

According to this table, the results obtained confirm the basic assumptions. All effects are positive.

III. Discussion and Conclusion

Our work is based on the measurement of reliability and validity of the TAM model in a Moroccan context. The study was conducted among Moroccan companies that provide cleaning and security services.

The basis of our case study is the questionnaire developed by Wong and Teo (2009). Indeed, our research has shown that the questionnaire items are correctly positioned in the four axes, also, the Cronbach's Alpha is greater than 0.7 so the constructs are reliable.

In conclusion, the results of the empirical study show that the TAM model is valid to explain and predict correctly the intention to use computer (digitalization) in the cleaning and security company.

This validation showed us that the cleaning and security companies accept to use the public procurement portal (PMP) as a tool for managing public procurement and consequently, we can say that these companies have a very positive perception of the digitalization applied by the Moroccan State.

The Limits :

Our research contains some limitations, they are the following:

- The choice of a reduced sample constituted only by cleaning and security companies makes it difficult to generalize the conclusions reached on all the users.

- The perception has a psychological aspect that cannot be evaluated by a quantitative approach (which we have adopted), and this can make the results unbalanced.

- The choice of an analysis model from 1989 to analyze a phenomenon in 2021 makes the study a little bit less satisfying.

References

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AN OVERVIEW ON MULTI-OBJECTIVE OPTIMIZATION AND DECISION MAKING

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In the field of operations research and optimization engineering using computer science, the core topic always was and still about finding optimal, acceptable and feasible solutions in complex spaces, and we mean by 'complex spaces' here spaces that are large and highly constrained and intricate: The real world is a perfect materialization of such space.

From an algorithmic perspective, finding optimal and acceptable solutions for real world problems is challenging. The introduction of the synonymous concepts of Multi-Objective Optimization and Decision Analysis (MODA) or the Multi-Criteria Decision Making (MCDM), which are and modern variants of optimization and decision making, as a key lever that takes into consideration the fact that in order to search for and find the optimum solution for a real world problem, we have to manage and combine between the features of Multi-Criteria Decision Aiding (MCDA) and Multi-Criteria Optimization (MCO), both form the combined science of making "good" decisions through the process of systematically studying and analyzing the different alternatives, options, choices, scenarios…

Optimization and decision making in real world problems often, if it's not always, involve conflicting parameters (criteria). For example, chosing a means and a path to travel from a point to another: it should be safe, fast, quick, cheap, comfortable, convenient, reliable… but you can't have it all! Or you are invited to engineer an industrial process that should be cost efficient, scalable, reusable, eco-friendly…

Finding THE optimum solution, usually referred as the 'ideal', where all objectives are at their optimal level, is the exception and not the rule. So the main idea is to reach a compromise, a win-win situation.

As a part of the scientific field of operations research and optimization engineering using computer science, we can define the Multi-Objective Optimization and Decision Analysis (MODA), or it synonym Multi-Criteria Decision Making (MCDM), as a multi-disciplinary field based on other scientific disciplines: Multi Criteria Decision Aiding (MCDA) and Multi-Criteria Optimization (MCO).

So the Multi-Objective Optimization and Decision Analysis (MODA), or the Multi-Criteria Decision Making (MCDM), deals with Multi-Criteria Decision Aiding (MCDA) and Multi-Criteria Optimization (MCO), or combinations of these.

We can define:

Multi-Criteria Decision Aiding (MCDA), also called Multi-Criteria Decision Analysis, as a scientific field that studies the evaluation of a finite number of alternatives based on multiple quantitative criteria. It supplies tools and methods to compare, evaluate and rank solutions.

Multi-Criteria Optimization (MCO), also called Multi-Criteria Design or Multi-Criteria Mathematical Programming, a scientific field that studies the search, in a usually large space, for optimal solutions while taking into consideration a number (multiple) of criteria and constraints. The size of the search space makes the inspection of all the solutions usually impossible.

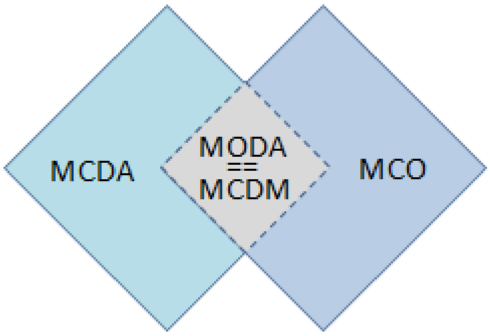

Figure – MODA/MCDM field situation

Due to the highly complex, constrained and intricate properties of its application field, the multi-objective optimization and decision-making problems present a rich algorithmic and mathematical structure, especially algorithms that use parallel computing in order to solve complicated application problems. In addition, their high relevance in various application fields in real life has recently led to an important rise in the interest aroused not only by the scientific community but also by different types of profiles: investors, influence groups and decision-makers (lobbyists, political parties, NGOs…) thus resulting in a very significant increase in research activities and the investment flows that comes with it.

In our work, we will therefore try to cover the subject from a broad perspective and understand its fundamentals from an algorithmic point of view to enlighten the reader and arouse his curiosity, thus prompting him to think further:

We will theoretically exhibit the foundations of Multi-Objective Optimization and their solutions methods as well as the tools for their performance-assessment. Also, we will provide an overview on formal processes and decision aid tools for conflicts resolutions.

Then we will present case studies from the literature, particularly the ones in real life application domains of decision making.

The main goal here is to introduce the reader to this fascinating field of interdisciplinary sciences and its foundations by providing a starting point through the exposition of its fundamentals and the different methods, techniques, applications as well as the terminology necessary to guide him in his quest for knowledge. References will of course be given to the reader in order to help him deepen his research and help point into more specialized topics.

Number of reasons makes the Multi-Criteria Optimization and Decision Making a thrilling and exciting scientific field of computer science and operations research, the most important one is the fact that different scientific fields are addressed in MCO and MCDA, so in MCDM.

Firstly, in order to build and develop the 1st fundamental layer methods for the MCO, we have to deal with elementary structural sciences, such as relational logic, algorithmic, operations research and numerical analysis. Eventually, some of the questions that we have to ask are:

How can we formalize a decision/optimization, problem in the right way?

What are the key differences between single-objective and multi-objective optimization?

How can we order the possible solutions? And what are the different types of rankings and orderings that are used in decision theory? And how are they related to each other?

What are the formal conditions that needs to be satisfied for a solution, or a number of solutions, to be optimal for an already given decision model or optimization problem?

How can we efficiently build optimal solutions providing algorithms, or at least ones that obtain approximations to them?

What is, from a geometrical stand of view, the right structure of solutions for problems with numerous optimal solutions? ...

Then when it comes to MCDA, and particularly making decisions in real world with all its constraints and parameters, these decisions will be made by humans, or machines that emulate human behavior, intelligence and even emotions, which are responsible for all the resulting impacts and consequences. In order to understand how the decisions are made, psychology of individuals and organizations needs to be studied. Some of the questions that might arise are:

What are our aims, goals and objectives? What makes it difficult to properly state them? How do we identify them? Can a formal process of doing so be supported?

Which strategies are used by people, humans and machines, when it comes to making a decision? And do they even use strategies? How satisfaction can be measured and what are the promising strategies to obtain such result?

What are the cognitive and behavioral aspects in decision making? How can decision making support systems be built in a proper way that take into consideration the cognitive and behavioral capabilities and limits of humans?

How do people, individuals and groups, come to decisions? What are the possible conflicts that can emerge through the decision making process? Can they be avoided and how? How can we deal with minority interests in a fair and democratic decision process? Is it possible to integrate all these aspects into formal decision processes and models?

Concretely, decisions are related to real world problems and are variable given the application field, so we may answer differently but specifically to these questions:

What is the set of possible alternatives, available options and best choices?

What are the means and the best practices to quantify a criterion or criteria? Experiments? Surveys? Evaluations? Are there any specific problems with these measurements given the field of application? What are the risks, dangers and costs? And how can we deal with them or even tackle them? Is there any possible uncertainties? (Off course yes!)

Given a field, or numerous fields, of application, what are the 'use-proof' decision processes? And what effects and implications have these best-practices on the engineering of decision support systems for the specific field or fields?

Are there specific optimization and decision support processes for specific existing problems? If so, what are the levels of acceptance and performances of these in practice?

What can we do about the reality of uncertain and distortion in today's world…

By this (tiny) list of questions, we are trying to have a panoramic perspective of the field, like some kind of bird's eye view, but the most important is that we are trying to arouse the reader's curiosity when it comes to the richness of the subject.

Practically, we will concentrate on the structural side of the Multi-Objective Optimization and Decision Making. However, we will also discuss the human-centric side of decision making, and in the other hand we will try to approach the problematic of Multi-Objective Optimization's utility and usability in application problems.

As a recap for the overview, it is important to note that the research field is very active and progresses have been recently made in every aspect of it. And in practice, it is wise and even mandatory to consider the human aspect of decision making, which can often be irrational or based on subjective criteria, but at the same time considers far more important things such as morality, ethics and deontology, which are key "criteria" that machines can't "understand" and emulate properly still to today.

Customer segmentation combining to Customer targeting : Overview

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Abstract*—Optimizing marketing campaigns is a very important process for companies to maximize their return on investment. However, this process goes through several stages in order to guarantee an optimal result. In this paper, we discuss two key phases of the marketing campaign optimization process, namely, customer segmentation and customer targeting. Customer segmentation involves segmenting customers into different similar groups in order to be able to make a prediction of the acceptability, by a given customer, of a given product offering. Customer targeting is mentioned just after this segmentation in order to determine which product to offer and to which customer in order to maximize the company's return on investment (ROI) while respecting some business constraints and based on the exit from the preceding phase which is the segmentation. This paper presents an overview on segmentation and the use of a machine learning algorithm for predicting customer behavior towards product offerings, and customer targeting using metaheuristics hybridization.

***Keywords*—Customer segmentation, Customer targeting Optimization, Machine Learning, Metaheuristic Hybridization, K-Means, Bat Algorithm**

I. INTRODUCTION

Machine Learning has demonstrated its segmentation efficiency in several areas; fraud detection, medical diagnostics, character recognition and others [1]. It is also efficient in customer segmentation which is an important concept in preparing and executing a marketing campaign plan and targeting the best customers in order to maximize the profit and ensure effective customer retention. Good customer segmentation is always seen as the key to a successful business strategy. This way of working allows business units to analyze and profile customers and produce an effective business plan. Customer segmentation can be defined as a model on which a company is based to offer new profitable offers capable of retaining customers and building loyalty; And in general, customers have similarity in their need as long as they have similar characteristics [1]. Various experimental researches have appeared annually in this field. However, it is always difficult to make comparisons between the different methods adopted to segment customers [2]. Previously, the concept of dividing customers by classifying it into similar groups was generally not based on the dependent / target variable. Marketing experts deduce that segmentation is a tool to achieve an objective, not an end in itself. As the majority of companies aim to maximize their ROI, Marketers concluded that the best customers will need to be separated from the rest of the customers through effective segmentation; hence the perceived popularity of clustering techniques while being based on a dependent variable [3]

II. MOTIVATION & METHODOLOGY

This paper is an overview of two previous works dealing separately with customer segmentation on the one hand, and customer targeting maximization in a marketing campaign on the other hand. The objective is to give an overview on a third work which will combine the two concepts mentioned above; the result of this possible study is to create a Framework which combine these two phases in a single process in order to produce a successful and optimal Marketing plan by maximizing the return on investment while taking into account the business constraints of a company. The first phase will be based on the RFM (Recency Frequency Monetary) segmentation model to predict customer behavior concerning the products that will be offered to them. A probability between 0 and 1 will be assigned to each segment. This prediction will make possible to calculate the value r_{ij} which is the expected revenue of the company from the offer of a product "j" to customer "i", where $r_{ij} = p_{ij}\, DFV_{ij}$.. DFV_{ij} is the return to the firm when customer i responds positively to the offer of product j, and p_{ij} is the probability that customer "i" respond positively to product "j" offer [4].

A. RFM segmentation

RFM segmentation is a widely used technique in identifying customer groups in Retail Marketing. It is a technique that aims to target specific groups with relevant characteristics of customer behavior and thus ensure a very high retention rate. By adopting an RFM segmentation technique, marketers will gain a very effective and in-depth understanding of their customers by analyzing three quantifiable factors:

- Recency: This parameter gives an idea of the time spent since the last purchase of a given customer. Sometimes, other variations may be considered other than a purchase, such as a customer's last visit to an online store. And generally, it is more likely that a customer, who has recently interacted with products, will respond positively to an offer.
- Frequency: This parameter gives an idea of the frequency of a customer's purchase (or reaction to an offer) during a specific period. The more frequent a customer is, the higher the probability of reacting to an offer.
- Monetary: this parameter gives an idea of the degree to which a customer is spending on products during a given period. Generally, Customers who spend the most should be treated in a special way compared to those who spend less.

The obvious first phase in creating an RFM model is assigning recency, frequency, and currency values to customers. Historically, raw data came from surveys and questionnaires sent to customers. Currently, a company's CRM or transactional databases make the task of data collection easier. The second step divides the set of customers into several groups of different levels for each Recency, Frequency, and Monetary dimension, so that each customer is assigned a level in each dimension. During the third step, groups of similar customers will be defined and will be sent specific types of communications according to the required RFM clusters. here are some examples to illustrate:

- Best Customers - This segment includes all customers who have transacted recently, do so multiple times, and spend larger amounts than the rest of the customers.
- New High Spend Customers - this second type of group concerns customers who have completed a single transaction, but with high value and very recently.
- Loyal active customers who spend the least - This is a type of customer who spends less but has been shopping recently and buys often.
- Churned Best Customers - This segment includes customers who buy frequently and with high amounts but their last transactions are long ago.

The fourth step actually goes beyond the RFM segmentation itself: crafting specific messaging that is tailored for each customer group. Based on the pattern of behavior of particular groups, RFM segmentation enables very effective communication with all customers and plays an important role in customer retention.

B. *K-Means algorithm*

One of the simplest unsupervised machine learning algorithms is K-Means clustering, which is widely used in solving clustering problems. The principle of this algorithm is to classify a set of data into a number of similar segments grouped into clusters. The letter k denotes a predefined number of clusters. Values belonging to the same cluster are very similar to each other on the one hand, and values between different clusters are very different on the other hand; This is precisely the role of K-Means in subdividing data into non-overlapping subsets. [5, 6].

The difficulty with K-Means lies in the fact that it takes precision to determine the cluster k, because the initialization of the cluster center can change in such a way as to cause unstable segmentation of the data [5, 6].

The K-Means clustering will be used to create customer segments based on the value of the probability that a customer will accept the offer of a given product while basing itself on quantitative FRM values for each customer.

C. *Customer Targeting*

Customer targeting is the act of reaching out to a portion of your customer list to re-engage them and drive sales. Popular tactics for these campaigns include direct mail and email. Social media and digital ads provide newer avenues to connect with your customer base with more speed and precision. In direct marketing campaigns, the optimization of targeted offers problem is a big business concern. The main goal is to maximize the company's profit by reaching the right customers. The main challenge faced by companies when advertising, is to configure properly a campaign by choosing the appropriate target, so it is guaranteed a high acceptance of users to advertisements [7].Direct Marketing is a tool that allows firms to promote their products directly to customers, and measure results quickly. One of the most important benefits of Direct Marketing is "Upgrading firm's loyalty strategies" in order to maximize the companies' Return on Investment. Nobibon et al. [4], and Cohen [8] before, presented the formulation of the product targeting problem as a mixed-integer programming (MIP) problem including more business constraints, and it can be rewritten as follows:

Given a set of m customers C = [c_1, c_2... c_m], and a set of n products' offers P = [p_1, p_2 …p_n], the objective is to maximize the Return on Investment under these business constraints:

- The corporate hurdle rate: each company defines its hurdle rate (HR) to make sure that the Return on Investment is equal, at least, to a value of HR.
- During the campaign, the budget of each product is limited.
- A limitation is imposed on the total number of products offered to each customer.
- And there is also a Minimum Quantity Commitment (MQC), which is the number of units of a product to be offered in order for that product to be part of the campaign. It means that no customer will receive an offer of a product which is not part of the campaign. If the product belongs to the proposed ones then at least Pj > 0 customers will receive an offer.

A solution is represented by a binary array $R_{|C|\times|P|}$, where C indicates the set of available costumers, and P represents the possible products to be used in the campaign. If a given cell $s_{i,j}$ |i ∈ C, j ∈ P is equal to "1" (true), the product j will be offered to the customer i; otherwise, the value would be "0" (false). There are two basic parameters of the customer lifetime value's computation; p_{ij} is the probability that customer i responds positively to an offer of product j, and DFV_{ij} is the return to the firm when customer i responds positively to the offer of product j [4, 10]. A basic formulation for the product targeting problem can be expressed, as modeled in [8], as:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(r_{ij}-c_{ij})x_{ij} - \sum_{j=1}^{m} fc_j y_j \qquad (1)$$

Where:

- r_{ij} is the expected revenue of the company from the offer of product "j" to customer "i", and $r_{ij} = p_{ij}\, DFV_{ij}$
- c_{ij} is the cost associated with the offer of product "j" to customer "i"
- M_i is the upper bound of products to offer to a customer "i"
- P_j is the minimum quantity commitment bound associated with product "j"
- B_j is the budget, in the campaign, allocated to the product "j"
- fc_j is the cost needed if a product j is used in the campaign
- O_j is the minimum quantity commitment bound associated with product j,
- HR is the hurdle rate specific to each company

The goal is to maximize the evaluation function given by Eq. (1), by finding the optimal combination of the two matrices X and Y:

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

Where:

$$x_{ij} = \begin{cases} 1 & \textit{If product j is offered to customer } \text{i}, \\ 0 & \text{Otherwise} \end{cases}$$

and,

$$y_j = \begin{cases} 1 & \textit{If product j is used in the campaign}, \\ 0 & \text{Otherwise} \end{cases}$$

The business constraints can be modeled as follows:

1. $$\sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}x_{ij} \geq (1+HR)\left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{j=1}^{n} fc_j y_j\right] \qquad (2)$$
2. $$\sum_{i=1}^{m} c_{ij}x_{ij} \leq B_j \quad \text{for each j = 1…, m} \qquad (3)$$
3. $$\sum_{j=1}^{n} x_{ij} \leq M_i \quad \text{for each i = 1…, m} \qquad (4)$$
4. $$\sum_{i=1}^{m} x_{ij} \leq m y_j \quad \text{for each j = 1…, n} \qquad (5)$$
5. $$\sum_{i=1}^{m} x_{ij} \geq P_j y_j \quad \text{for each j = 1…, n} \qquad (6)$$
6. $$x_{ij}, y_{ij} \in \{0,1\} \quad \text{for each i =1…m and j=1 …, n} \qquad (7)$$

D. Metaheuristic Hybridization :

In general, we use a hybridization of metaheuristics to solve very complex optimization problems in a very optimal response time. The first technique of relay hybridization means applying a set of metaheuristics in a sequential way (one after the other); each metaheuristic uses the output of the previous one as input. The second technique uses several metaheuristics but in a parallel way; it is a cooperative optimization model; each agent (method) launches a search in a search space while acting with the other agents [10]. This type of hybridization is more efficient in terms of response time and can give better results depending on the problem treated. Four classes are derived from this hierarchical taxonomy:

- Low-level Relay Hybrid (LRH): It is a class of algorithm characterized by the integration of a metaheuristic into a Single Solution-based metaheuristic (Talbi 2009).
- Low-level teamwork hybrid (LTH): It is a very popular type of hybridization and has proven to be effective in many types of optimization problems. In this type of hybridization, a metaheuristic is combined with a P-metaheuristic (a type of memetic algorithm). [10]
- High-level Relay Hybrid (HRH): In this type of hybridization, two or more metaheuristics are executed in an independent and sequential manner. A metaheuristic can be initialized with another metaheuristic to guarantee the generation of a feasible and good quality initial solution; this positively impacts the performance of the optimization process [10].
- High-level Teamwork Hybrid (HTH) : Several algorithms perform searches autonomously and in parallel with other metaheuristics while cooperating to optimize the result. In any case, the performance of the HTH is guaranteed to be at least the same level as any metaheuristics involved

E. Dataset :

To measure the effectiveness of using metaheuristics, Nobibon et al [4] created a randomly generated dataset as follows:

- Cost c_{ij} is randomly generated from the set {1,2,3}.
- The return of the firm r_{ij} is an integer generated randomly between 0 and 16.
- The corporate hurdle rate HR can take 5%, 10%, and 15% values.
- In this paper, there are three different values of the number of customers: 100 customers for small category S, 1000 customers for medium category M, and 10,000 customers for large category L.
- For each category of customers, we have two different numbers of product n: 5 and 10 products.
- For each combination (category of customer and number of products), a minimum-quantity

commitment bound P_j is randomly generated between $\left[\frac{\sum_i M_i}{n}\right]$ and $\left[2\frac{\sum_i M_i}{n}\right]$.

- The fixed cost fc_j is generated randomly between $\frac{O_j}{2m(1+HR)}\sum_i\left[p_{ij}-(1+HR)c_{ij}\right]$ and $\frac{O_j}{m(1+HR)}\sum_i\left[p_{ij}-(1+HR)c_{ij}\right]$
- For budget B_j, three values are adopted: the two values of $\left[O_j\frac{\sum_i c_{ij}}{m}\right]$ $\left[2\frac{\sum_i M_i}{n}\frac{\sum_i c_{ij}}{m}\right]$ and a random integer between $\left[O_j\frac{\sum_i c_{ij}}{m}\right]$ and $\left[2\frac{\sum_i c_{ij}}{n}\right]$.
- The upper bound M_i is selected between 1 and n/5.

However, the objective of the Framework that will are proposing in a future work, is to combine the customer segmentation with the optimization of the customer targeting in a single Framework to ensure an optimal and successful marketing campaign. In this case, the value of r_{ij} will not be random but rather calculated from the value p_{ij} which is the probability that a customer i responds positively to a product offer j. This value is a prediction calculated with an unsupervised Machine learning algorithm such as K-Means.
Once the r_{ij} values are computed, we proceed to an optimization of the fitness function in equation (1) while using an HRH hybridization. In [7], we combined the Genetic Algorithm (GA) to generate an initial solution that is feasible and of good quality, followed by the Binary Bat Algorithm (BBA) which is characterized by its speed of convergence. In order to optimize the BBA, we modified the BA in such a way as to escape stagnation in a local optimal as proposed by [11]. Figure Fig.1 shows the efficiency of the hybridization adopted as well as the modification made to the BBA.

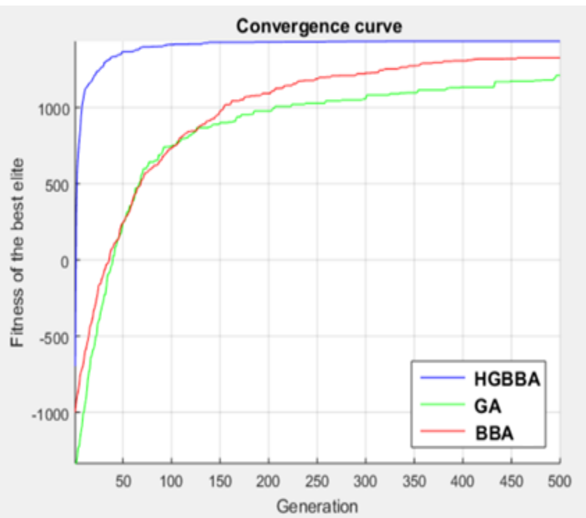

Fig. 1: Example of execution of three different algorithm for customer targeting optimization

III. Conclusion

This paper presents an overview of two works done with the same objective of optimizing a marketing campaign and maximizing the return on investment of a company. The two works are complementary and it is very wise to combine them in order to approach the problem of optimizing a company's marketing plan; the first phase, which is segmentation, is essential to make a prediction about a customer's behavior and the probability that he will respond positively to a given product offer. The second phase is there to complete the process and to launch an optimization of the fitness function with the objective of determining which product to offer and to which customer in order to reach a maximum profit. The segmentation phase is handled by building an RFM model and predicting the probabilities of customer response to offers through a Machine Learning algorithm which is the K-Means algorithm. The customer targeting optimization phase is just a classical non-linear optimization problem that is treated by a metaheuristic hybridization to guarantee a better result and with the best performances.

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Numerical solution of intuitionistic fuzzy differential equations by Diagonally implicit block backward differentiation formulas

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Abstract

In this work, the diagonally implicit block backward differentiation formulas (DIBBDF) is developed for solving intuitionistic fuzzy differential equations (IFDEs) under the interpretation of Hukuhara differentiability. The applicability and accuracy of the proposed method has been demonstrated by an example. It is clearly shown that the proposed method obtains good numerical results and suitable for solving IFDEs.

Index Terms

Numerical solution, Block method, Diagonally implicit.

I. Introduction

Fuzzy set theory was proposed by Zadeh in 1965 [21], and it has been applied in various fileds. The principe of the fuzzy set is an extension of the characteristic function taking the value of zero or one to the membership function which can take any value from the interval $[0, 1]$. Therefore, the membership function is only a single-valued function, which cannot be used to express the evidences of support and objection simultaneously in many situations. As the fuzzy set cannot be used to make totally explicit all the information or data in such a problem, it confronts a variety of limits in actual applications. For that reason, Atanassov in 1983 and 1986 [1], [2] extended the fuzzy set characterized by a membership function to the intuitionistic fuzzy sets (IFSs), which is characterized by a membership function, a non-membership function, and a degree of uncertainty. Then, the IFSs can describe the fuzzy characters of things in a more detailed and comprehensive way, which is found to be more effective in dealing with vagueness and uncertainty. The intuitionistic fuzzy set has already achieved great success in theoretical research and real applications [18]–[20].

Differential equations with uncertainty plays serve as mathematical models in many fields such as science, physics, economics, psychology, defense and demography. This type of differential equations is called intuitionistic fuzzy differential equations (IFDEs). The topic of intuitionistic fuzzy differential equations (IFDEs) have been rapidly growing in recent years. The first attempt to treat IFDEs has been done in [14]. In recent years, the authors have focused on existence-uniqueness results for intuitionistic fuzzy solutions of some types of IFDEs. They have defined the concept of intuitionistic fuzzy solutions and introduced conditions for existence and uniqueness results using different techniques [6], [7], [9], [10], [12], [16]. The still-standing problem in the theory of intuitionistic fuzzy differential equations is to find implementable numerical methods. Much more effort has been made in this direction as well. There are some applications of numerical methods are introduced in [3]–[5], [8], [11], [13], [17]. In this paper, intuitionistic fuzzy differential equation is solved numerically by diagonally implicit block backward differentiation formulas under the interpretation of Hukuhara differentiability.

The paper is organized as follows: In Section 2, some basic definitions and the IFDE is defined. The derivation and the implementation of implicit block backward differentiation formulas for IFDEs are presented Section 3. The results of these numerical methods are discussed in section 4. The final section is the conclusion.

II. Preliminaries

A. Notations and definitions

Throughout this paper, $(\mathbb{R}, B(\mathbb{R}), \mu)$ denotes a complete finite measure space.
Let us $P_k(\mathbb{R})$ the set of all nonempty compact convex subsets of $\mathbb{R}$.
we denote by

$$\mathbb{F}_1 = \mathrm{IF}(\mathbb{R}) = \{\langle u, v\rangle \,:\, \mathbb{R} \to [0,1]^2, |\forall\, x \in \mathbb{R}\ 0 \leq u(x)+v(x) \leq 1\}$$

An element $\langle u, v\rangle$ of $\mathbb{F}_1$ is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0,\ x_1 \in \mathbb{R}$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
(ii) u is fuzzy convex and v is fuzzy concave.
(iii) u is upper semi-continuous and v is lower semi-continuous
(iv) $supp\langle u, v\rangle = \mathrm{cl}\{x \in \mathbb{R} \ :|\ v(x) < 1\}$ is bounded.

so we denote the collection of all intuitionistic fuzzy number by $\mathbb{F}_1$.

Definition 2.1: Let $\langle u,v\rangle$ an element of $\mathbb{F}_1$ and $\alpha\in[0,1]$, we define the following sets :

$$\Big[\langle u,v\rangle\Big]_l^+(\alpha)=\inf\{x\in\mathbb{R}\mid u(x)\geq\alpha\},$$
$$\Big[\langle u,v\rangle\Big]_r^+(\alpha)=\sup\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^-(\alpha)=\inf\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\},$$
$$\Big[\langle u,v\rangle\Big]_r^-(\alpha)=\sup\{x\in\mathbb{R}\mid v(x)\leq 1-\alpha\}$$

Remark 2.1:

$$\Big[\langle u,v\rangle\Big]_\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^+(\alpha),\Big[\langle u,v\rangle\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle\Big]^\alpha=\Big[\Big[\langle u,v\rangle\Big]_l^-(\alpha),\Big[\langle u,v\rangle\Big]_r^-(\alpha)\Big]$$

A Triangular Intuitionistic Fuzzy Number (TIFN) $\langle u,v\rangle$ is an intuitionistic fuzzy set in $\mathbb{R}$ with the following membership function u and non-membership function v :

$$u(x)=\begin{cases}\frac{x-a_1}{a_2-a_1} & \text{if } a_1\leq x\leq a_2\\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2\leq x\leq a_3\\ 0 & \text{otherwise}\end{cases}$$

$$v(x)=\begin{cases}\frac{a_2-x}{a_2-a_1'} & \text{if } a_1'\leq x\leq a_2\\ \frac{x-a_2}{a_3'-a_2} & \text{if } a_2\leq x\leq a_3'\\ 1 & \text{otherwise}\end{cases}$$

where $a_1'\leq a_1\leq a_2\leq a_3\leq a_3'$ and $u(x),v(x)\leq 0.5$ for $u(x)=v(x),\quad\forall x\in\mathbb{R}$
This TIFN is denoted by $\langle u,v\rangle$=$\langle a_1,a_2,a_3;a_1',a_2,a_3'\rangle$ where,

$$[\langle u,v\rangle]_\alpha=[a_1+\alpha(a_2-a_1),a_3-\alpha(a_3-a_2)] \quad (1)$$
$$[\langle u,v\rangle]^\alpha=[a_1'+\alpha(a_2-a_1'),a_3'-\alpha(a_3'-a_2)] \quad (2)$$

Theorem 2.1: ([15])
d_∞ define a metric on IF_1.

Theorem 2.2: The metric space (IF_1,d_∞) is complete.

Remark 2.2: If $F:[a,b]\to\mathbb{F}_1$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[0,1]$ then

$$F(t)=F(t_0)+\int_{t_0}^t F'(s)ds$$

Definition 2.2: Let $\langle u,v\rangle$ and $\langle u',v'\rangle\in IF_1$, the H-difference is the IFN $\langle z,w\rangle\in IF_1$, if it exists, such that

$$\langle u,v\rangle-\langle u',v'\rangle=\langle z,w\rangle\Longleftrightarrow\langle u,v\rangle=\langle u',v'\rangle+\langle z,w\rangle$$

Definition 2.3: A mapping $F:[a,b]\to IF_1$ is said to be Hukuhara derivable at t_0 if there exist $F'(t_0)\in IF_1$ such that both limits:

$$\lim_{\Delta t\to 0^+}\frac{F(t_0+\Delta t)-F(t_0)}{\Delta t}$$

and

$$\lim_{\Delta t\to 0^+}\frac{F(t_0)-F(t_0-\Delta t)}{\Delta t}$$

exist and they are equal to $F'(t_0)=\langle u'(t_0),v'(t_0)\rangle$, which is called the Hukuhara derivative of F at t_0.

B. Intuitionistic fuzzy Cauchy problem

In this section, we consider the initial value problem for the intuitionistic fuzzy differential equation

$$\begin{cases}x'(t) &= f(t,x(t)),\quad t\in I\\ x(t_0) &= \langle u_{t_0},v_{t_0}\rangle\in IF_1\end{cases} \quad (3)$$

where $x\in IF_1$ is unknown $I=[t_0,T]$ and $f:I\times IF_1\to IF_1$ and $x(t_0)$ is intuitionistic fuzzy number.
Denote the $\alpha-$ level set

$$[x(t)]_\alpha=\Big[[x(t)]_l^+(\alpha),[x(t)]_r^+(\alpha)\Big]$$
$$[x(t)]^\alpha=\Big[x(t)]_l^-(\alpha),[x(t)]_r^-(\alpha)\Big]$$
$$[x(t_0)]_\alpha=\Big[[x(t_0)]_l^+(\alpha),[x(t_0)]_r^+(\alpha)\Big]$$
$$[x(t_0)]^\alpha=\Big[x(t_0)]_l^-(\alpha),[x(t_0)]_r^-(\alpha)\Big]$$

and

$$[f(t,x(t))]_\alpha=\Big[f_l^+(t,x(t);\alpha),[f_r^+(t,x(t);\alpha)\Big]$$
$$[f(t,x(t))]^\alpha=\Big[f_l^-(t,x(t);\alpha),[f_r^-(t,x(t);\alpha)\Big]$$

Sufficient conditions for the existence of an unique solution to Eq. (3) are:

1) Continuity of f
2) Lipschitz condition: for any pair $(t,x_1),(t,x_2)\in I\times\mathbb{F}_1$, we have
$$d_\infty\Big(f(t,x_1),f(t,x_2)\Big)\leq Kd_\infty(x_1,x_2) \quad (4)$$
where $K>0$ is a given constant.

Theorem 2.3: [6] Let us suppose that the following conditions hold:

(a) Let $R_0=[t_0,t_0+p]\times\overline{B}(\langle u,v\rangle_{t_0},q)$, $p,q\geq 0$,$\langle u,v\rangle_{t_0}\in IF_1$ where $\overline{B}(\langle u,v\rangle_{t_0},q)=\{\langle u,v\rangle\in IF_1: d_\infty(\langle u,v\rangle,\langle u,v\rangle_{t_0})\leq q\}$ denote a closed ball in IF_1 and let $f:R_0\longrightarrow IF_1$ be a continuous function such that $d_\infty\big(f(t,\langle u,v\rangle),0_{(1,0)}\big)\leq M$ for all $(t,\langle u,v\rangle)\in R_0$.
(b) Let $g:[t_0,t_0+p]\times[0,q]\longrightarrow\mathbb{R}$ such that $g(t,0)\equiv 0$ and $0\leq g(t,x)\leq M_1$, $\forall t\in[t_0,t_0+p], 0\leq x\leq q$ such that $g(t,x)$ is non-decreasing in u and g is such that the initial value problem
$$x'(t)=g(t,x(t)),x(t_0)=x_0. \quad (5)$$
has only the solution $x(t)\equiv 0$ on $[t_0,t_0+p]$
(c) We have $d_\infty\Big(f\big(t,\langle u,v\rangle\big),f\big(t,\langle z,w\rangle\big)\Big)\leq g(t,d_\infty(\langle u,v\rangle,\langle z,w\rangle))$, $\forall(t,\langle u,v\rangle)$, $(t,\langle z,w\rangle)\in R_0$ and $d_\infty(\langle u,v\rangle,\langle z,w\rangle)\leq q$. Then the intuitionistic fuzzy initial value problem
$$\begin{cases}\langle u,v\rangle' &= f(t,\langle u,v\rangle),\\ \langle u,v\rangle(t_0) &= \langle u,v\rangle_{t_0}\end{cases} \quad (6)$$

has an unique solution$\langle u,v\rangle \in C^1[[t_0,t_0+r],B(x_0,q)]$ on $[t_0,t_0+r]$ where $r=\min\{p,\frac{q}{M},\frac{q}{M_1},d\}$ and the successive iterations

$$\langle u,v\rangle_0(t)=\langle u,v\rangle_{t_0},$$
$$\langle u,v\rangle_{n+1}(t)=\langle u,v\rangle_{t_0}+\int_{t_0}^{t} f(s,\langle u,v\rangle_n(s))ds$$

converge to $\langle u,v\rangle(t)$ on $[t_0,t_0+r]$.

C. *Interpolation of intuitionistic fuzzy number*

The problem of interpolation for intuitionistic fuzzy sets is as follows:
Suppose that at various time instant x information $f(x)$ is presented as intuitionistic fuzzy set. The aim is to approximate the function $f(x)$, for all x in the domain of f. Let $x_0<x_1<\ldots<x_n$ be $n+1$ distinct points in $\mathbb{R}$ and let $\langle u_0,v_0\rangle,\langle u_1,v_1\rangle,\ldots,\langle u_n,v_n\rangle$ be $n+1$ intuitionistic fuzzy sets in IF_1. An intuitionistic fuzzy polynomial interpolation of the data is an intuitionistic fuzzy-value continuous function $f: I\longrightarrow IF_1$ satisfying:

- $f(x_i)=\langle u_i,v_i\rangle$
- If the data is crisp, then the interpolation f is a crisp polynomial.

A function f which fulfilling these condition may be constructed as follows. For each $Y=(y_0,y_1,\ldots,y_n)\in\mathbb{R}^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_Y such that

- $P_Y(x_i)=y_i, \quad i=0,1,\ldots,n$
- $P_Y(x)=\sum_{i=0}^n y_i\Big(\prod_{i\neq j}\frac{x-x_j}{x_i-x_j}\Big)$

According to the extension principle, we can write the membership and non-membership function $f(x)$ for each $x\in\mathbb{R}$ as follows:

$$\mu_{f(x)}(t)=\begin{cases}\displaystyle\sup_{\substack{y_0,y_1,\ldots,y_n\\ t=P_{y_0,y_1,\ldots,y_n}(x)}}\ \min_{i=0,1,\ldots,n}\mu_{u_i}(y_i) \\ \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t)\neq\emptyset \\ 0 \qquad \text{otherwise}\end{cases}$$

where μ_{u_i} is the membership function of u_i, and

$$\nu_{f(x)}(t)=\begin{cases}\displaystyle\inf_{\substack{y_0,y_1,\ldots,y_n\\ t=P_{y_0,y_1,\ldots,y_n}(x)}}\ \max_{i=0,1,\ldots,n}\nu_{v_i}(y_i) \\ \text{if } P^{-1}_{y_0,y_1,\ldots,y_n}(t)\neq\emptyset \\ 1 \qquad \text{otherwise}\end{cases}$$

where ν_{v_i} is the non-membership function of v_i.
Let $J_i^+(\alpha)=[\langle u_i,v_i\rangle]_\alpha$, $J_i^-(\alpha)=[\langle u_i,v_i\rangle]^\alpha$ for any $\alpha\in[0,1]$, $i=0,1,...,n$ and $[f(x)]_\alpha$, $[f(x)]^\alpha$ the upper and lower α-cuts of $\langle u_i,v_i\rangle$ and $f(x)$ respectively. Hence,

$$\begin{aligned}[f(x)]_\alpha &= \big\{t\in\mathbb{R}\ \mid\ \mu_{f(x)}(t)\geq\alpha\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, y_0,y_1,\ldots,y_n:\mu_{u_i}(y_i)\geq\alpha,\\ &\qquad i=0,\ldots,n \text{ and } P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, Y\in\prod_{i=0}^n J_i^+(\alpha)\\ &\qquad : P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\end{aligned}$$

and

$$\begin{aligned}[f(x)]^\alpha &= \big\{t\in\mathbb{R}\ \mid\ \nu_{f(x)}(t)\leq 1-\alpha\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, y_0,y_1,\ldots,y_n:\nu_{v_i}(y_i)\leq 1-\alpha,\\ &\qquad i=0,\ldots,n \text{ and } P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\\ &= \big\{t\in\mathbb{R}\mid \exists\, Y\in\prod_{i=0}^n J_i^-(\alpha)\\ &\qquad : P_{y_0,y_1,\ldots,y_n}(x)=t\big\}\end{aligned}$$

Finally, for each $x\in\mathbb{R}$ and all $t\in\mathbb{R}$ is defined by $f(x)\in IF_1$ by

$$f(x)(t)=\Big(\sup\big\{\alpha\in(0,1]|\exists Y\in\prod_{i=0}^n J_i^+(\alpha):P_Y(x)=t\big\},\\ 1-\sup\big\{\alpha\in(0,1]|\exists Y\in\prod_{i=0}^n J_i^-(\alpha):P_Y(x)=t\big\}\Big)$$

where $Y=(y_0,y_1,\ldots,y_n)\in\mathbb{R}^{n+1}$
The interpolation polynomial can be written level set wise as

$$\begin{aligned}[f(x)]_\alpha &= \{y\in\mathbb{R}:y=P_{y_0,y_1,\ldots,y_n}(x),\\ &\qquad y_i\in[\langle u_i,v_i\rangle]_\alpha \quad i=0,\ldots,n\},\quad \text{for}\quad \alpha\in(0,1]\end{aligned}$$

and

$$\begin{aligned}[f(x)]^\alpha &= \{y\in\mathbb{R}:y=P_{y_0,y_1,\ldots,y_n}(x),\\ &\qquad y_i\in[\langle u_i,v_i\rangle]^\alpha \quad i=0,\ldots,n\},\quad \text{for}\quad \alpha\in(0,1]\end{aligned}$$

But, from Lagrange interpolation formula, we have

$$[f(x)]_\alpha=\sum_{i=0}^n \ell_i(x)J_i^+(\alpha)$$

and

$$[f(x)]^\alpha=\sum_{i=0}^n \ell_i(x)J_i^-(\alpha)$$

where $\ell_i(x)$ represents the Lagrange polynomials.
When the data $\langle u_i,v_i\rangle$ presents as triangular intuitionistic fuzzy numbers, values of the interpolation polynomial are also triangular intuitionistic fuzzy numbers. Then $f(x)$ has a particular simple form that is well suited to computation. Denote $J_i^+(\alpha)=[a_i^+(\alpha),b_i^+(\alpha)]$ and $J_i^-(\alpha)=[a_i^-(\alpha);b_i^-(\alpha)]$. Then the upper end point of $[f(x)]_\alpha$ is the solution of the optimization problem :

$$\text{Maximize}\quad P_{y_0,y_1,\ldots,y_n}(x)\quad \text{subject to}\quad a_i^+(\alpha)\leq y_i\leq b_i^+(\alpha)\\ i=0,1,\ldots,n$$

It follows that the optimal solution is

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) \geq 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \end{cases}$$

and the lower end point is obtained as the value of the interpolation polynomial associated to points

$$y_i = \begin{cases} b_i^+(\alpha) & \text{if} \quad \ell_i(x) < 0 \\ a_i^+(\alpha) & \text{if} \quad \ell_i(x) \geq 0 \end{cases}$$

Similarly the upper and lower end point of $[f(x)]^\alpha$ can be obtained.
Hence if $\langle u_i, v_i \rangle$ is an intuitionistic fuzzy number, for all i then also $f(x)$ is such an intuitionistic fuzzy number for each x. More precisely, if $\langle u_i, v_i \rangle = \langle u_i^l, u_i^c, u_i^r, v_i^l, u_i^c, v_i^r \rangle$ and $f(x) = \big\langle f_l(x), f^c(x), f_r(x), f^l(x), f^c(x), f^r(x) \big\rangle$, then we will have,

$$\begin{aligned} f_l(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)u_i^l + \sum_{\ell_i(x)<0} \ell_i(x)u_i^r \\ f_r(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)u_i^r + \sum_{\ell_i(x)<0} \ell_i(x)u_i^l \\ f^c(x) &= \sum_{i=0}^{n} \ell_i(t)u_i^c \\ f^l(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)v_i^l + \sum_{\ell_i(x)<0} \ell_i(x)v_i^r \\ f^r(x) &= \sum_{\ell_i(x)\geq 0} \ell_i(x)v_i^r + \sum_{\ell_i(x)<0} \ell_i(x)v_i^l \end{aligned}$$

III. Formulation of Diagonally Implicit Block Backward Differentiation Formulas

The fully implicit block backward differentiation was derived using Lagrange polynomial, $P_k(t)$ of degree k which interpolates the values $x_n, x_{n-1}, \ldots, x_{n-k+1}$ of a function f at interpolating points $t_n, t_{n-1}, t_{n+1}, \ldots, t_{n-k+1}$ in terms of Lagrange polynomial defined as follows:

$$P(t) = \sum_{j=0}^{k} \ell_{k,j}(t) f(t_{n+1-j})$$

where $\ell_{k,j}(t) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{(t-t_{n+1-i})}{(t_{n+1-j}-t_{n+1-i})}$, for $j = 0, 1, \ldots, k$.
Next, we defined $s = \frac{t-t_{n+1}}{h}$ and replace $f(t, x)$ by polynomial (6) which interpolates only the values x_{n-1}, x_n and x_{n+1} at the interpolating points $t_{n-1}, t_n, \ldots, t_{n+1}$ to compute x_{n+1}.

$$\begin{aligned} P(t) = \; & \tfrac{(t_{n+1}+sh-t_n)(t_{n+1}+sh-t_{n+1})}{(-h)(-2h)} x(t_{n-1}) \\ & + \tfrac{(t_{n+1}+sh-t_{n-1})(t_{n+1}+sh-t_{n+1})}{(h)(-h)} x(t_n) \\ & + \tfrac{(t_{n+1}+sh-t_{n-1})(t_{n+1}+sh-t_n)}{(2h)(h)} x(t_{n+1}) \end{aligned}$$

Thus, differentiating the resulting polynomial once with respect to s at the point $t = t_{n+1}$ and evaluating at $s = 0$ gives the following corrector formula for first point as follows

$$x_{n+1} = -\frac{1}{3}x_{n-1} + \frac{4}{3}x_n + \frac{2}{3}hf(t_{n+1}, x_{n+1}) \qquad (7)$$

Then, we interpolate the values x_{n-1}, x_n, x_{n+1} and x_{n+2} and at the interpolating points $t_{n-1}, t_n, t_{n+1}, \ldots, t_{n+2}$ to compute x_{n+2} and differentiating the resulting polynomial once with respect to s at the point $t = t_{n+2}$ and substituting $s = 0$ yields

$$x_{n+1} = \frac{2}{11}x_{n-1} - \frac{9}{11}x_n + \frac{18}{11}x_{n+1} + \frac{6}{11}hf(t_{n+2}, x_{n+2}) \quad (8)$$

Let the exact solutions

$$[X(t_n)]_\alpha = \Big[[X(t_n)]_l^+(\alpha), [X(t_n)]_r^+(\alpha)\Big]$$

$$[X(t_n)]^\alpha = \Big[[X(t_n)]_l^-(\alpha), [X(t_n)]_r^-(\alpha)\Big]$$

be approximated by

$$\begin{aligned} [x(t_n)]_\alpha &= \Big[[x(t_n)]_l^+(\alpha), [x(t_n)]_r^+(\alpha)\Big] \\ [x(t_n)]^\alpha &= \Big[[x(t_n)]_l^-(\alpha), [x(t_n)]_r^-(\alpha)\Big] \end{aligned} \qquad (9)$$

at t_n, $0 \leq n \leq N$
The solutions are calculated by grid points at

$$t_0 < t_1 < t_2 < \ldots < t_N = T, \quad h = \frac{T - t_0}{N}, \\ t_n = t_0 + nh, \quad n = 0, 1, \ldots N$$

By using formula (7) and (8), we configure the general intuitionistic fuzzy DIBBDF for approximates solutions as follows:

$$\begin{cases} [x(t_{n+1})]_l^+(\alpha) = -\frac{1}{3}[x(t_{n-1})]_l^+(\alpha) + \frac{4}{3}[x(t_n)]_l^+(\alpha) \\ \qquad + \frac{2}{3}hf_l^+(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_r^+(\alpha) = -\frac{1}{3}[x(t_{n-1})]_r^+(\alpha) + \frac{4}{3}[x(t_n)]_r^+(\alpha) \\ \qquad + \frac{2}{3}hf_r^+(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_l^-(\alpha) = -\frac{1}{3}[x(t_{n-1})]_l^-(\alpha) + \frac{4}{3}[x(t_n)]_l^-(\alpha) \\ \qquad + \frac{2}{3}hf_l^-(t_{n+1}, x(t_{n+1}); \alpha) \\ [x(t_{n+1})]_r^-(\alpha) = -\frac{1}{3}[x(t_{n-1})]_r^-(\alpha) + \frac{4}{3}[x(t_n)]_r^-(\alpha) \\ \qquad + \frac{2}{3}hf_r^-(t_{n+1}, x(t_{n+1}); \alpha) \end{cases}$$

and

$$\begin{cases} [x(t_{n+2})]_l^+(\alpha) = \frac{2}{11}[x(t_{n-1})]_l^+(\alpha) - \frac{9}{11}[x(t_n)]_l^+(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_l^+(\alpha) + \frac{6}{11}hf_l^+(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_r^+(\alpha) = \frac{2}{11}[x(t_{n-1})]_r^+(\alpha) - \frac{9}{11}[x(t_n)]_r^+(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_r^+(\alpha) + \frac{6}{11}hf_r^+(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_l^-(\alpha) = \frac{2}{11}[x(t_{n-1})]_l^-(\alpha) - \frac{9}{11}[x(t_n)]_l^-(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_l^-(\alpha) + \frac{6}{11}hf_l^-(t_{n+2}, x(t_{n+2}); \alpha) \\ [x(t_{n+2})]_r^-(\alpha) = \frac{2}{11}[x(t_{n-1})]_r^-(\alpha) - \frac{9}{11}[x(t_n)]_r^-(\alpha) \\ \qquad + \frac{18}{11}[x(t_{n+1})]_r^-(\alpha) + \frac{6}{11}hf_r^-(t_{n+2}, x(t_{n+2}); \alpha) \end{cases}$$

IV. Example

In this section, one set of intuitionistic fuzzy initial value problem is tested for the purpose of validating the difference in numerical results. The test problem and solution are listed below.

Example 4.1: Consider the intuitionistic fuzzy initial value problem

$$\begin{cases} x'(t) &= x(t) \text{ for all } t \in [0,T] \\ x_0 &= \Big((\alpha-1, 1-\alpha), (-2\alpha, 2\alpha)\Big) \end{cases} \tag{10}$$

Then, we have the following parametrized differential system:

$$\begin{cases} [x(t)]_l^+(\alpha) = & (\alpha-1)\exp(t) \\ [x(t)]_r^+(\alpha) = & (1-\alpha)\exp(t) \\ [x(t)]_l^-(\alpha) = & -2\alpha\exp(t) \\ [x(t)]_r^-(\alpha) = & 2\alpha\exp(t) \end{cases}$$

Therefore the exact solutions are given by

$$[X(t)]_\alpha = \Big[(\alpha-1)\exp(t), (1-\alpha)\exp(t)\Big]$$

$$[X(t)]^\alpha = \Big[-2\alpha\exp(t), 2\alpha\exp(t)\Big]$$

which at $t=1$ are

$$[X(1)]_\alpha = \Big[(\alpha-1)\exp(1), (1-\alpha)\exp(1)\Big]$$

$$[X(1)]^\alpha = \Big[(-2\alpha\exp(1), 2\alpha\exp(1)\Big]$$

Comparison of results of the the DIBBDF method and Runge-Kutta Nyström method in [11] for $h=0.2$ and $t=1$:

	Exact	
α	$([X]_l^+, [X]_r^+)$	$([X]_l^-, [X]_r^-)$
0	(-2.718281828,2.718281828)	(0,0)
0.2	(-2.174625462,2.174625462)	(-1.087312731,1.087312731)
0.4	(-1.630969097,1.630969097)	(-2.174625462,2.174625462)
0.6	(-1.087312731,1.087312731)	(-3.261938194,3.261938194)
0.8	(-0.543656365,0.543656365)	(-4.349250925,4.349250925)
1	(0,0)	(-5.436563656,5.436563656)

	DIBBDF		RK-Nyström	
α	$([x]_l^+, [x]_r^+)$	$([x]_l^-, [x]_r^-)$	$([x]_l^+, [x]_r^+)$	$([x]_l^-, [x]_r^-)$
0	(-2.718251136,2.718251136)	(0,0)	(-2.717509377,2.717509377)	(0,0)
0.2	(-2.174600909,2.174600909)	(-1.087300454,1.087300454)	(-2.174007501,2.174007501)	(-1.087003750,1.087003750)
0.4	(-1.630950681,1.630950681)	(-2.174600909,2.174600909)	(-1.630505626,1.630505626)	(-2.174007501,2.174007501)
0.6	(-1.087300454,1.087300454)	(-3.261901363,3.261901363)	(-1.087003750,1.087003750)	(-3.261011252,3.261011252)
0.8	(-0.543650227,0.543650227)	(-4.349201818,4.349201818)	(-0.543501875,0.543501875)	(-4.348015003,4.348015003)
1	(0,0)	(-5.436502273,5.436502273)	(0,0)	(-5.435018754,5.435018754)

α	Error in RK-Nyström	Error in DIBBDF
0	3.8622×10^{-4}	1.5345×10^{-5}
0.2	4.6347×10^{-4}	1.8415×10^{-5}
0.4	5.4071×10^{-4}	2.1484×10^{-5}
0.6	6.1796×10^{-4}	2.4553×10^{-5}
0.8	6.9520×10^{-4}	2.7622×10^{-5}
1	7.7245×10^{-4}	3.0691×10^{-5}

V. Conclusion

In this paper, we have applied a new block method for numerical solution of intuitionistic fuzzy differential equations. The proposed method obtains better accuracy compared to the existing method in terms of error. We can conclude that DIBBDF is one of the efficient methods for solving intuitionistic fuzzy differential equations.

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Digital marketing in the era of artificial intelligence

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***Abstract*— Artificial intelligence is a hope for some but remains as a danger for others.**

It is both a progress and a nightmare that begins to worry the experts of the sectors by estimating that the arrival and the establishment of the intelligent robots will be a destructive fact of employment and will initiate the replacement of the human by a conscious artificial intelligence. [1]

Artificial intelligence is gradually becoming part of our daily lives and is giving way to transformations that are sometimes fictional and sometimes apparent reality. Like other sectors (health, automotive, industry, astronomy, art ...) AI also affects the marketing sector in general and digital marketing in particular. We can see it clearly: for years now, the web giants, or the so-called GAFAM (Google, Amazon, Microsoft ...) have been implementing artificial intelligence tools in their processes, products and services.

In this paper, we aim to explore various applications of artificial intelligence (AI) in digital marketing. Artificial intelligence has been studied by many researchers using exploration as a secure basis for research. Artificial intelligence has been the subject of investigation since the 1950s, starting with when Alan Turing's question sparked controversy: "Can machines think? "

From there, artificial intelligence plays a central role in the various fields of society. Thanks to artificial intelligence, « automatons recognize articulated speech and understand texts written in natural language, cars drive themselves, robots wage war in place of men, some scientists even seek to conquer death by determining the mechanisms of aging ...

Not only this, but most dimensions of intelligence - except perhaps humor – are the object of rational analyzes and reconstructions with computers, and in addition, machines go beyond cognitive faculties in most areas...» [2]

Considered as a 'game changer' in the marketing industry and beyond, especially content marketing, artificial intelligence has become so powerful that for example, a machine wrote a screenplay that was turned into a science-fiction and the possibilities of evolution never end and who knows, very soon the machines will take over the role of humans.

In this perspective, content marketing intervenes, which becomes MARKETING itself, as a key variable of the research project.

The approach to content marketing has changed over the years, we increasingly hold take into account consumer preferences and trends to satisfy the target.

The world of content is changing at high speed and marked by the arrival of news technologies and alignment with new trends. Content marketing is no longer a project that we carry out in parallel, but it is a project that we explore full time. This paper can serve as a guide for a content manager and a content creator and help him know, how can AI contribute to the evolution of content creation.

***Keywords*— Artificial intelligence, Digital marketing, robots, IA, machine learning, deep learning.**

I. INTRODUCTION

To make sense of artificial intelligence is to give it a direction, a meaning and an explanation. [3]

Giving a direction is based on the objectives of artificial intelligence in relation with the sectors that benefit from its development. [3]

To give a meaning is to emphasize the complementarity between humans and intelligent systems. [3]

And to give an explanation is to lead a reflection on this artificial consciousness and to explain the development of

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artificial intelligence, its infinite potential, its use, its power and its limits. [3]

AI affects all sectors, but if there is one sector that is a natural candidate for AI, it is digital marketing, media and communication.

And that’s what we will explore during this overview : how digital marketing is evolving in the era of artificial intelligence, and more specifically, how AI is transforming the sector of artificial intelligence, giving some examples of practices, uses and applications of AI in digital marketing.

II. ARTIFICIAL INTELLIGENCE AND DIGITAL MARKETING

The concept of artificial intelligence was born in the 1950's. It is a notion that appeared thanks to the mathematician Alan Turing, in his book Computing Machinery and Intelligence, where he raised the question of bringing a form of intelligence to machines. [4]

Artificial intelligence hides behind its operation devices with a common denominator and that is machine learning, or what is also called automatic learning.

The latter is known as a set of techniques that give an algorithm the ability to learn without having been programmed explicitly for each of the instructions.

It is a technology that allows storing a large amount of data in a brain or virtual neural network and also dominates as a sub-discipline, deep learning, where the machine is inspired by the human brain.

Artificial intelligence is making its way into almost every sector and is being implemented in many fields of application.

The evolution of this algorithmic technology is very noticeable and allows us to solve more and more complex problems that we would have thought were reserved to human intelligence.

Starting with a simple chatbot, AI is becoming more and more specialized and efficient and contributes to decision making.

AI has also been introduced and strongly in the field of digital marketing. It is worth noting that 84% of marketing leaders believe that AI is essential to help them achieve their goals. [5]

AI guarantees time savings, accuracy and performance on tasks performed. It has become the marketer's main ally and allows him not only to automate certain tasks but also to obtain results in record time.

In this way, marketing teams can focus on all what is strategic and entrust artificial intelligence with all that is time-consuming tasks where the machine can be autonomous. [6]

Artificial intelligence also promises a better ROI, while boosting marketing actions. A recent study by Accenture estimates that companies that invest in AI can expect to increase their revenues by about 30% over the next 4 years. [7]

A survey by Forbes Insights and Quantcast makes a strong case that artificial intelligence technologies are having a real impact on any company's goals.

Specifically, in numbers: for 52% of marketers surveyed, applications that rely on machine learning help increase sales and 49% say they help launch new products successfully. [8]

III. AI GUARANTEES EVER MORE PERSONALIZED AND EFFICIENT MARKETING

The use of artificial intelligence in digital marketing ensures a 100% personalized customer experience. [9]
AI algorithms tend towards further personalization and offer a unique experience, through the right content, offered at the right time.
Users of a digital platform and consumers of a product or service increasingly favor the personalized approach, where the value proposition is customer-centric, or user-centric.
Artificial intelligence tools analyze through the collected data and propose what will meet the user's need. Personalized recommendations from Amazon, Netflix or Facebook are the best example. The AI technologies used rely on deep learning and Big Data systems to improve the user experience. These tools do not stop there but also generate reports to give the company the chance to review or adapt its digital marketing strategy.

Artificial intelligence also contributes to the emergence of content marketing. [9] Several platforms exist to offer functions related to predictive content technology. This means that they not only evaluate the relevance of the themes composing your content strategy but also give you a taste of how your audience will react to your content. This feeds the machine's algorithm and allows it to propose adjustments, based on competitors' content and the feedback on your content. In the end, the tool proposes new topics, adequate to the needs of your users and guaranteeing a very good user experience, a good referencing of the brand and a large reach on the web.

Artificial intelligence is also designed to automate repetitive tasks and improve marketers' productivity [9]. This is called marketing automation and there are many examples of platforms. We can quote Mailchimp for example which allows the automation of e-mailing campaigns and thus distribute the content to a targeted and qualified audience.

IV. What about content marketing ?

Content marketing is known to be a discipline of reference in the sphere of the web, content marketing is the marketing strategy which aims to attract customers.

For a successful communication strategy, we must provide content that is not only relevant but of quality, which will arouse the interest of customers and seduce prospects to convert them later. Content should be informative, should bring good advices, should be entertaining, and ultimately contain emotional traits. And these four elements form what we call the characteristics or ingredients of good content.

And when we are talking about content marketing, we're not only articles or texts but any type of media content such as visuals, videos, infographics, graphs, market studies, webforms ...

The relationship between the concepts of artificial intelligence, digital marketing and content marketing is very important, because the evolution of one can impact the emergence of the other.

Artificial intelligence has been studied by many researchers using exploration as safe bases for research. Artificial intelligence has been investigated since the 1950s, starting from when Alan Turing's question created a controversy: "Can machines think? " From then on, artificial intelligence has played a central role in the various fields of society. Thanks to artificial intelligence, "automatons recognize articulated speech and understand texts written in natural language, cars drive themselves, robots make war instead of men, some scientists are even trying to defeat death by determining the mechanisms of aging ... Most dimensions of intelligence - except perhaps humor – are the subject of analysis and rational reconstructions with computers, but moreover the machines are outperforming cognitive faculties in most domains ... " [2]

The machine has become a key component in the foreseeable future. Researchers have not only revealed that it is having a significant impact but that it is increasingly performing human tasks that could never be outsourced.

Recently, more attention has been paid to the influence of artificial intelligence on our life. This research sheds new light on the influence of the machine on the content creation, literary, before focusing on content marketing. The current research aims to understand, analyze and explore the effects of the machine on content, to compare machine-generated content with human- generated content and to see if the reader is able to differentiate between the two contents, and if the content produced by a machine has the same effect as human content on its reader. The results should subsequently make an important contribution to the field of content creation.

Could artificial intelligence impact content marketing? This issue analyzes the relationship between two variables: Artificial intelligence and content marketing. Today, new evolutions and emergences result from artificial intelligence which covers more and more a very vast field and almost infinite fields of application. Touching on machine translation, chatbots, autonomous cars, optimization of commercial prospecting, smart building, HR processes and the list goes on, artificial intelligence is in the process of disrupting all sectors and all fields.

Following the evolution of artificial intelligence, always in perpetual mutation, great dazzling progress has been reported. Artificial intelligence has overtaken humans in some aspects and who knows, maybe and very soon, humans will be completely replaced and instead of natural toughts, we will face artificial intelligence.

By impacting all sectors, artificial intelligence, a scientific subject, has not failed to bring about major changes in marketing by becoming very associated with it.

Considered a `` game changer " in the marketing sector and more particularly in content marketing, artificial intelligence has become so powerful that, for example, a machine even wrote a script that was transformed into science fiction film and the possibilities of evolution are numerous and who knows, very soon, machines will take over the role of humans.

In this perspective, content marketing comes into play, which becomes MARKETING in itself, as a key variable in the research project. The approach to content marketing changes over the years, we increasingly take into account consumer preferences and trends to satisfy the target. The world of content is evolving at high speed and is marked by the arrival of new technologies and alignment with new trends. Content marketing is no longer a project that we carry out in parallel, but it is a project that we explore full time.

We are currently seeing content serving customer satisfaction, snack content that leaves room for longer content which is slow content, fairly diverse content, personalization and authenticity of rendering and several other criteria that make content marketing a strong intermediary, paving the way for a two-way dialogue between the company and its target audience.

Content marketing of course has its ingredients that humans have taken the time to master and experiment with for a long time.

The turning point of the web has highlighted the content of the brand as part of a hybrid marketing strategy, and the man has been able to put content processes into practice in innovation strategies, he has also known well personalize the content and align it with the different trends of the moment. Humans use a generic approach based on an analysis combining content and consumer needs and try through content to transmit knowledge with all its challenges and developments. Humans try to keep a balance between content marketing and B2B information and also set up a targeted and personalized brand content strategy for the benefit of companies. All this following well quantified and detailed analyzes of course, but the essential point is the human intuition that monetizes the content and makes it useful for marketing purposes.

That being said, it seems, artificial intelligence is more efficient and effective in terms of results.

V. Some Theories That Can Be Discussed

A. *Roko's Basilisk Theory [10]*

Roko's Basilisk Theory or what is also called the fear of artificial intelligence is known as a thought experiment proposed in 2010 by user Roko on the Less Wrong site [11]

As some have called it, it was the most terrifying thought experiment of all time, which made techno-futurists very frightened [12]. It is an evil and divine form of artificial intelligence, so dangerous that if we see it, or even think about it too much, we will spend the rest of eternity screaming in our torture chamber.

One part tends not to believe this, while another believes the existence of this artificial intelligence which takes over the control of all humanity, thus reducing humans to slaves. It is a theory that has known deep reflection in scientific circles in computer science.

It all started in 2010 when Roko wrote an article on the Less Wrong site, explaining that the evolution of technology will certainly lead to the appearance of a powerful artificial intelligence on earth. This same intelligence, and when it appears, will decide to punish all humans who do not encourage it to emerge and evolve rapidly. That is to say that humans, and when this artificial intelligence will appear, are obliged to put their financial resources at the service of the emergence of this super artificial intelligence. Otherwise, anyone who has not contributed will have to suffer an eternal artificial hell.

In general, this theory comes to demonstrate the irrational fear of certain people in front of Artificial Intelligence. And this is one of the many proposals to describe the behavior of the technological singularity that will occur in the future, to explain its principles and to qualify it as artificial intelligence whose power exceeds human understanding.

B. *General Golem Theory*

Jewish cabalistic tradition records the existence of a clay statue made by Rabbi Loew, better known as the "Maharal of Prague," towards the end of the 16th century. Like contemporary computers, this machine came to life when you passed a message behind your teeth.

Usually, she went about her daily household chores, like a zealous and diligent servant.

Many legends have run around this extraordinary statue. According to one of them, Rabbi Loew forgot, on a Saturday, a day of prayer, to remove the message from behind the Golem's teeth and the Golem started to fidget, scream and frighten neighbors, while his master fulfilled his holy duties in the synagogue. Back home, Rabbi Loew is said to have destroyed her work for fear that she would start taking unfortunate initiatives again. " [2]

C. *Content Shock [13]*

It is a theory that favors quantity over quality [14]. It is mainly based on our limited ability to consume content while its production is increased. Content being the very essence of the internet, the user finds himself in the middle of a lot of information which he has no time to read, nor to see and even less to understand. Although there are different ways of expressing it, we quickly understand that this phenomenon is due to the overproduction of matter. One could even translate it by a saturation, a bulimia and an overdose of information. You have to be present at all costs, and by doing that, Content Shock decides to prioritize the quantity of content over its quality.

VI. Some Applications of Artificial Intelligence in Digital Marketing

A. *Creating and Generating Content*

The current challenge for brands and businesses is the way they sell their services. It's a whole process that comes into play of course, from the attraction phase to the conversion and loyalty phase.

What could guarantee a good understanding of the brand is the content addressed to the customers.

This is to say that content today is the main element that allows to position or not the brand on the web, vis-à-vis its competitors.

Brands often lack inspiration, and it is in this sense that artificial intelligence tools tend to produce the most relevant content for users.

Artificial intelligence algorithms allow you to create and generate content, either for your website or to generate your analytical reports. Not just any kind of content, but a base is already there.

Some scientific researches deal with what is called automated content, especially in the field of journalism and many questions are raised about the effect of generated content on the psychology of readers. [15]

To generate content, the machine analyzes the data of several contents to produce one that is similar. Intelligent tools exist and are already used by Forbes to create news.

To do this, the machine uses an existing template, fills in the blanks and adds keywords to give the reader the impression that the content is written by an expert writer.

We can take the example of the Washington Post which uses an artificial intelligence named Heliograf to produce articles dealing with sports news, politics and many other topics. The objective is to offer a diverse content to adapt to the different tastes of the readers.

BBC, the New York Times and Reuters implement this technology as well.

B. *Digital and online Advertising*

In terms of this application, the example is clear. Facebook as well as Google's advertising platforms use artificial intelligence tools to deliver targeted ads to a specific audience, with a pre-defined budget and with an idea of the expected results.

It works by analyzing users' information (their interests, their demographic data ...) to offer them the product that will meet their needs. In this way, the brand or company reaches its target directly, without wasting time, but optimizing its expenses and resources.

C. *Sales prediction*

Many companies have always wanted to have an estimate of the sales they will make over a given period. Artificial intelligence saves them the effort and helps them predict their sales. AI tools now analyze, in a very fine way, external variables that can contribute to the prediction of sales in a given sector.

For example, we can cite the searches made on the digital devices of an advertiser, the result and frequency of searches allows the brand to organize itself in terms of production quantity, storage ... and everything related to the control of resources and logistics related to it.
If we take the example of a book during the phase of its release on the market. Knowing the number of likes, shares and comments on the announcement of the publication of this book, on one of the social media networks, will allow the publishing house to estimate the production volume.

D. *Customer/prospect scoring and targeting*

Artificial Intelligence is proving itself and can now identify the prospects of a company or a business. Not all prospects are identified, but those considered to be the most interesting ones that can be converted into future customers. Thanks to artificial intelligence algorithms, we identify the prospects with the highest conversion rate. This is called the "look-alike" method.

The result is to identify an audience that has the criteria of the business's clientele, having the same profile, sharing the same interests, interested in the same product categories ...

This allows the business to optimize acquisition costs, save time and resources.

E. *Personalization (zoning and editorial content)*

In order to optimize the user experience on a platform for example, artificial intelligence tools tend to test different zoning on a page, in order to identify the most optimal experience. We start with test, learn and optimize and this is what allows us to guarantee an optimized content.

We take here the example of mailchimp, an e-mailing tool that ensures the distribution of newsletters and after sending, we can analyze certain KPIs, such as the opening rate, on which areas users click the most and thus which content is the most consulted.

F. *Chatbots (or conversational agents)*

This has become the trend in the banking sector for example. The majority of Moroccan or French banks' websites contain a chat bot that allows to humanize the customer experience, to answer the customers' questions and to guarantee the contact. NLP (Natural Language Processing) and NLU (Natural Language Understanding) are used to create conversations between man and machine.

And customers feel they have a real person in front of them, who is interacting with them and answering all their questions, in a very short period of time, compared to the waiting lines on the phones and in the agencies.

G. Listening and analyzing social noise

Artificial intelligence tends to analyze the digital voice of people on social networks.

This creates an opportunity for the company to monitor and have an idea of its image conveyed on social networks.

This is also called social listening, where the brand listens to its community through human language recognition techniques. In other words, artificial intelligence tools analyze what is said and written on the web and social networks about a given brand. The added value is that even the tone of the comments is scrutinized to allow the brand to identify influencers and act in case of misunderstanding.

VII. CONCLUSION

Maybe innovation has allowed the company to change ! Its processes and trying to deliver an automated content, through artificial intelligence tools, to save time, energy and money. But the most important thing that the company should take into consideration is that every customer is different and we absolutely can't deliver the same content to everyone. And beside all of that, will an automated content attract the customer in the same way of a content written by a specialist, a content creator or a writer ?

The writer spend time to analyze the need of its readers, he settles on their primary profile, then develop the right content, and this help him get the right message and get the overall messaging strategy pinned down. To win the attention of the primary customers, the content should be sufficiently clear. It should speak the lexicon of the customers and understand them well enough to articulate the solution the company is presenting to their problem. Will a machine, producing an automated content, be able to understand the customer before proposing him the content he is looking for ? Creating powerful, compelling messaging depends entirely on knowing who we are addressing.

The content writer needs to identify what affects the psychology of their readers and consumers and mainly know what kind of language they use.

The content writer needs to find the nuances that will let him get inside the mind of his readers. And of course, each content in the marketing field needs to be focused on "adding value " [16]

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The Impact of COVID-19 Pandemic on the Civil Engineering sector VS The various economic and financial measures : Moroccan case

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***Abstract*—Since the beginning of the Coronavirus pandemic, many companies have had to adapt their working methods to several sectors such as the construction industry. Like many countries, the Moroccan economy is based mainly on sector construction. The problem of sources of financing of this sector is essential, Relaunching the machine with a national program on 100% state financing, for a wide range of projects in the field of public infrastructure and equipment, is it still possible in this period of austerity? The purpose of this study is to determine by a cross-sectional study based on an online questionnaire, the effectiveness of the various sources of financing in favor of construction sector managers and how they see the future of civil engineering and the construction industry.**

Keywords—Finance, Civil Engineering, Construction, Morocco, COVID-19

I. Introduction

Engineering and construction industries are used to cyclical downturns, however, the strength and the speed which COVID-19 has struck is exceptional. Thus, many projects are being canceled or delayed. Supply chains are under threat. Subcontractor and employee labor health is a concern, and there are practical defies on social distancing on construction zones. Companies that have had to decrease workers might find the future skills and availability of those workers unclear. Also, due to many contracting and construction companies operate without risk of insolvency or substantial capital [1]–[3].

In Morocco, the COVID-19 pandemic, like all countries in the world is still going through an unprecedented health challenge with economic consequences [4]. Because of the inherent nature of the business activity, some industries may unwittingly prosper under the pandemic. Although the continuation of construction work has always been authorized by the government and administrative authorities during a state of a health emergency, the fact remains that several factors have contributed to the decline of the construction sector caused mainly by the unavailability of material due to the temporary closure of many production sites or, for those whose activity has always been maintained, were faced to the insufficient funds. Moreover, the ban on travel between the different cities amplifies the abyss by making it impossible to move workers who were willing to occupy construction sites amid a health crisis. According to the figures provided by the National Federation of Building and Public Works (FNDBT), the construction sector recorded a significant decrease of 75% in the Building sector and a 60% decrease in Public Works [5] .

Thus, the purpose of this study is to determine by a cross-sectional study based on an online questionnaire, the effectiveness of the various sources of financing in favor of construction sector managers and how they see the future of civil engineering and the construction industry.

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A. Civil Engineering Sources Of Financing

1) Public financing

Public investment is still under pressure and it will be difficult to escape the current austerity situation. Morocco will have to choose between operating and investment spending and promote certain sectors to the detriment of others; Relaunch the machine with national programs on 100% state funding, for a wide range of projects in the field of public infrastructure and equipment [6] .

2) National Federation of Building and Public Works (FNDBT)

The FNBTP requires the interest-free deferral of loan repayment deadlines to banks and leasing companies as announced by the Monitoring Committee of all loans including those granted in the context of pledging contracts. At the same time, FNBTP wants to systematically introduce an advance of 10% to 20% for all public procurement, in addition to a national preference of 15% for all public procurement. The Federation wishes to encourage the banks to play their role and accompany the companies in difficulty and not to refuse to capitalize on the public markets [7] .

3) Tax policy and administration (Economic Monitoring Committee)

Tax relief could be extended to small- and medium-sized enterprises and regions or industries most affected. Such relief measures could include income tax credits, rate cuts, and exemptions, deadline extension and deferral, extended loss carry-back rules, or limiting advance tax payments [8] .

4) « Relaunch Very Small Entreprise (VSE) » and « Damane Relaunch »

To deal with COVID19, the economic vigilance committee established two mechanisms of loan guaranteed by the Fund Guarantee Center (CCG: Caisse Centrale de Garantie) " Relaunch VSB " and "Damane Relaunch"(Figure1). These two mechanisms help to finance the need for business funds with a maximum interest rate of 4%, which represents a key rate Bank Al-Maghrib+ 200 points. Repayment of these credits can be extended over seven years with a forbearance of two years [9].

The product "Relaunch VSB ", intended to guarantee loans from very small businesses with a turnover of less than 10 million DH, has benefited no less than 10,756 companies, totaling 2.4 billion DH of loans granted, for a commitment volume of 2.3 billion DH.

Concerning the " Damane Relaunch " product, it deployed for small, medium and large sized companies

with a turnover exceeding 10 million DH, has benefited 4,427 companies. Thus, this mechanism covered a close to 20 billion DH, for 17.4 billion DH of commitments [9] .

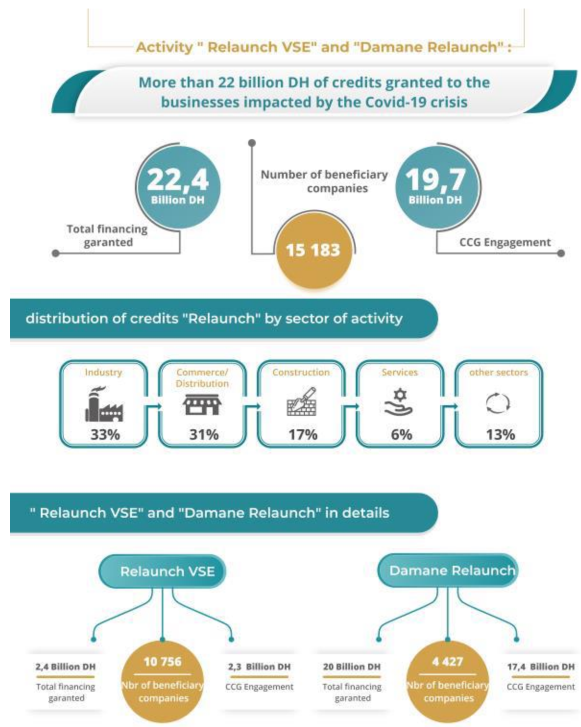

Fig.1. Activity « Relaunch VSE » and « Damane Relaunch »

B. Purpose Of The Study

The purpose of this study is to determine by a cross-sectional study based on an online questionnaire, the effectiveness of the various sources of financing in favor of construction sector managers and how they see the future of civil engineering and the construction industry.

II. METHODOLOGY

a) Study design

Given the new and unexplored nature of the research problem, our study is a cross-sectional investigation that was based on a self-administered online questionnaire. The data will be analyzed used the SPSS software.

b) Questionnaire design and data collection

The questionnaire was designed using Google Forms, which is a free electronic medium offered by Google, that can voluntarily collect information through the designed questionnaire.

The participated population in our study were selected based on a purposive sampling methodology. In fact, we have identified through a few digital forums of entrepreneurs and engineers that we have contacted directly in private to solicit their participation. The choice of the participants was based on their activities in civil engineering. In total, 76 people have participated in our survey. Among those participants, 38 are managers, 18 are directors of the project, 14 are engineers and 6 are financial directors.

III. RESULTS AND DISCUSSION

Our results show that 22% of participants work in Small and Medium Entreprise *(*SME*)* and 78% in Small Entreprise (VSE). Concerning the impact of COVID19 on their activities, 96% indicated that their company was affected by COVID-19, and 4 % was not (as the design civil engineers were able to work from home). Furthermore, our study illustrates that 61,1% of projects are being delayed or canceled, 55,6% of supply chains are under threat, 69,4 % of employee and subcontractor labor health is a concern, and there are practical challenges around social distancing on construction sites and 56,9 % of companies that have had to furlough workers (Figure2). The effects of the covid-19 crisis in the civil engineering sector showed in our results can be illustrated as Domino Theory: Each effect is the result of the last one[10].

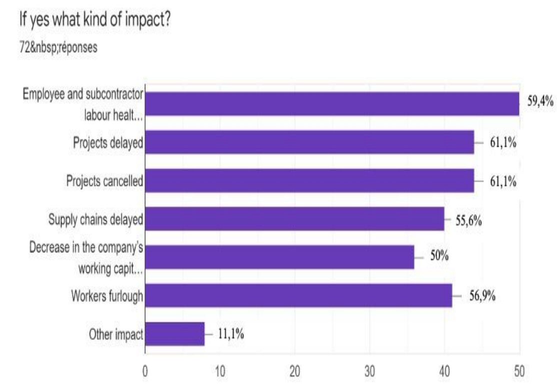

Fig.2. Impact type on the civil engineering sector.

The result of Figure 3 presents the answers to the question: what the sources of funding are to overcome the effects of the covid19 pandemic.

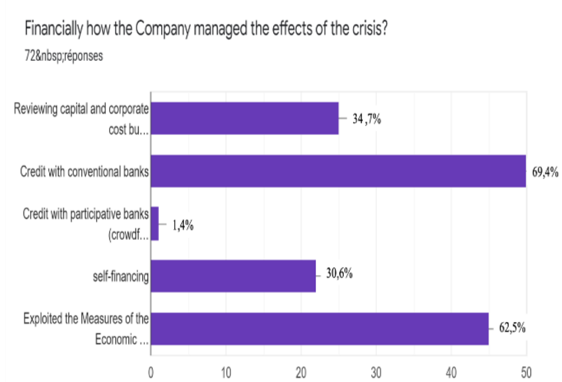

Fig.3. Finance sources.

By analyzing Figure 3, we note the absence of participatory banks (crowdfunding) to support construction companies during this period of crisis. However, the main support is based mainly on the traditional bank loans (69,4%) through the two products dedicated to overcoming this pandemic: credit daman relaunch 22,4 Billion DH distributed (the Engineering and construction organizations benefited with17%) and credit daman oxygen 7,2 Billion DH distributed (the Engineering and construction organizations benefited with 22%) [11].

In addition, 62,5% was simply using the Economic Monitoring Committee measures, this committee has set up different fiscal, economic and social measures to overcome this pandemic crisis. The fiscal measure such as the postponement of the filing of tax returns, suspension of tax audits and Notices to third-party. The economic measure like Suspension of the payment of social charges, postponement of payment of bank without penalties or surcharges and social measurements are such as compensation of 2000 DH is paid to the benefit of employees reporting to employers affiliated to the Moroccan national social security fund with economic difficulties.

Other funding sources with no significant difference are reviewing capital and corporate cost budget and self-financing with 34,7% and 30,6% respectively as revealed in Figure 2.

Taken together, the results of this study will provide us information if a single source of finance will be sufficient to close this COVID-19 financing gap. Furthermore, it will give us responses across all sources of the finance mix to "stop the bleeding" and

possibly avoiding a collapse of financing for the development of the construction sector.

Many Engineering and construction organizations are confronting a financial shock, with an especially significant impact on their cash flow [12]. In the short term, they should conduct an extensive project-by-project forecast and source government financial support. In addition, management teams should consider contractual terms, the recoverability of receivables in a site shutdown, and the inevitable inefficiencies created by remote working and on-site distancing restrictions. In the medium term, many organisations will need to renegotiate lending arrangements and raise new equity.

At the point when the pandemic ends, engineering and construction companies will face a new world. The marketplace will change, as some national governments will be eager to invest in infrastructure to jump-start their recovery, and others may face new resource limitations. Cities will need to serve more residents who work from home — likely leading to greater investments in telecom and smart city initiatives.

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Application of First Order-Richardson Method to Systems of Linear Equations with Fuzzy Data

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***Abstract*—In this paper, we will applied the Richardson iterative method for solving the fuzzy linear equations. In addition, we explain the efficiency of suggested method by solving a numerical example as a fuzzy version of examples from classical circuit analysis.**

***Index Terms*—Fuzzy linear system; Fuzzy number; Richardson iterative method; Circuit Analysis.**

I. Introduction

Systems of linear equations find many real world applications in different areas scientifics, In this background, several iterative method are developed because of their simplest implementation and their lest cost of computational complexity. The fuzzy set term first appeared in 1965 when Professor Lotfi A. Zadeh of Berkeley University, USA, published an article entitled Fuzzy Sets. He has since achieved many major theoretical advances in the field and was quickly accompanied by many researchers developing theoretical work.

In this paper, we think over a fuzzy linear system with an arbitrary fuzzy number in parametric form and with a crisp coefficient matrix. Consequently, there is hugely of the numerical iterative methods for the resolution of the fuzzy linear systems such as: Gauss-Seidel (G-S), Jacobi (J), and Successive Over Relaxation (SOR) iterative method[4,5]. Next, a first order Richardson iterative method is presented for the fuzzy linear systems nonsingular.

This manuscript will begin with a fundamental construct of fuzzy number operation is bringed In Section 2. In Section 3, the principal Section of the paper, a Richardson approach is for solving crisp and fuzzy linear system. The suggested idea is shown by solving a numerical example in the Section 4. Finally,we will conclude our paper with a small conclusion given in Section 5.

II. Preliminaries

A. *Fuzzy sets on the real line - Fuzzy numbers*

We begin this section with some preliminary results for fuzzy sets :

• The fuzzy sets on the real line $\mathbb{R}$ are caracteresed by theirs membership functions $\tilde{u} : \mathbb{R} \to [0,1]$ ([2], [9], [8] and [14]).

• we define by $[\tilde{u}]^\alpha := \{x \in \mathbb{R}/\tilde{u}(x) \geq \alpha\}$ and $[\tilde{u}]^0 := cl(\{x \in \mathbb{R}/\tilde{u}(x) > 0\})$ The α-cut ($\alpha \in [0,1]$) on the fuzzy set $\tilde{u}$ on $\mathbb{R}$

whither the closure of the set X is denotes by cl(X).

• A fuzzy set $\tilde{u}$ is called convex if

$$\tilde{u}(\lambda x + (1-\lambda)y) \geq min(\tilde{u}(x), \tilde{u}(y)), \quad x, y \in \mathbb{R}, \quad \lambda \in [0,1]$$

Let $\mathbf{F}(\mathbb{R})$ denotes de family of all fuzzy sets on $\mathbb{R}$.
We tell that $\tilde{u} \in \mathbf{F}(\mathbb{R})$ is a fuzzy number, if and only if : its membership function is defined as : (1) $\tilde{u}$ is normal.
(2) $\tilde{u}$ is convex.
(3) $\tilde{u}$ is upper semicontinuous.
(4) The α-cut $[\tilde{u}]^0$ is compact.

The set of all possible fuzzy numbers $\tilde{u}$ shall be called the fuzzy-number power set $\mathcal{F}(\mathbb{R})$ with the property $\mathcal{F}(\mathbb{R}) \subset \mathbf{F}(\mathbb{R})$.

• Particular fuzzy numbers :
i)A popular fuzzy number is triangular fuzzy number $\tilde{a}$ defined by a triplet $[a, \alpha, \beta]$. As well as

$$\tilde{a}(x) = \begin{cases} 1 + \frac{x-a}{\alpha} & ; \quad a - \alpha \leq x \leq a, \\ 1 + \frac{a-x}{\beta} & ; \quad a \leq x \leq a + \beta. \end{cases}$$

ii) A trapezoidal fuzzy number $\tilde{a}$ can be expressed as $[a_L, a_U, \alpha, \beta]$ and its membership function is defined as :

$$\tilde{a}(x) = \begin{cases} 1 + \frac{x-a_L}{\alpha} & ; \quad a_L - \alpha \leq x \leq a_L, \\ 1 & ; \quad a_L \leq x \leq a_U. \\ 1 + \frac{a_U-x}{\beta} & ; \quad a_U \leq x \leq a_U + \beta. \end{cases}$$

• The definition of addition, soustraction and scalar multiplication on $\mathcal{F}(\mathbb{R})$ are as follows :
For $\tilde{u}$,$\tilde{v} \in \mathcal{F}(\mathbb{R})$ and $\lambda \geq 0$,

$$(\tilde{u} + \tilde{v})(x) := \sup_{x_1, x_2 \in \mathbb{R}/x_1+x_2=x} min(\tilde{u}(x_1), \tilde{v}(x_2)) \tag{1}$$

$$(\tilde{u} - \tilde{v})(x) := \sup_{x_1, x_2 \in \mathbb{R}/x_1-x_2=x} min(\tilde{u}(x_1), \tilde{v}(x_2)) \tag{2}$$

$$(\lambda\tilde{u})(x) := \begin{cases} \tilde{u}(x/\lambda) & \text{if} \quad \lambda \in \mathbb{R} - \{0\}, \\ 1_{\{0\}}(x) & \text{if} \quad \lambda = 0. \end{cases} \tag{3}$$

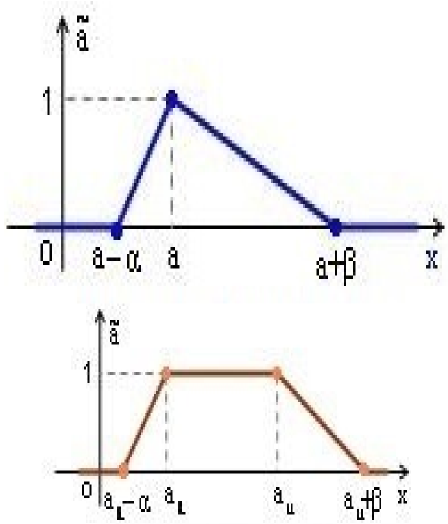

Fig. 1: Triangular and Trapezoidal Fuzzy Number

where $1_{\{0\}}$ is an indicator.
we denote by K the set of all nonempty compact subset of R and by K_C the subsets of K consisting of nonempty convex compact sets. Recall that

$$\rho(A,B)=\min_{a\in A}\|x-a\|$$

is the distance of the point $x\in R$ from $A\in K$, and that the Hausdorff separation $\rho(A,B)$ of $A,B\in k$ is defined as $\rho(A,B)=\max_{a\in A}\rho(a,B)$.
The Hausdorff metric d_H on K is defined by

$$d_H(A,B)=max\{\rho(A,B),\rho(B,A)\}$$

.
The prolongement of this metric, the Hausdorff metric on $\mathcal{F}(\mathbb{R})$ is defined by:

$$d_\infty(\tilde{u},\tilde{v})=\sup\{d_H(\tilde{u}[\alpha],\tilde{v}[\alpha]):0\le r\le 1\}$$

B. Fuzzy linear systems

Definition 2.1:
The $n\times n$ linear system of equations

$$\begin{aligned}a_{11}x_1+a_{12}x_2+.....+a_{1n}x_n&=y_1,\\ a_{21}x_1+a_{22}x_2+.....+a_{2n}x_n&=y_2,\\ a_{31}x_1+a_{32}x_2+.....+a_{3n}x_n&=y_1,\\ \\ a_{n1}x_1+a_{n2}x_2+.....+a_{nn}x_n&=y_n,\end{aligned}\tag{4}$$

These sets of equations will be defined by the following matrix:

$$AX=Y\tag{5}$$

.
where the coefficient matrix $A=a_{ij}$, $1\leqslant i,j\leqslant n$ is a crisp $n\times n$ matrix and $y_i\in E^1$, $1\leqslant i\leqslant n$ This system is called a fuzzy linear system (FLS).

C. Solution of fuzzy linear systems

For arbitrary $x=(\underline{x}(r),\bar{x}(r))$, $y=(\underline{y}(r),\bar{y}(r))$ and real number $k>0$, the usual arithmetic operations of fuzzy numbers, can be represented like this

1) $\underline{(x+y)}(r)=\underline{x}(r)+\underline{y}(r)$
2) $\overline{(x+y)}(r)=\bar{x}(r)+\bar{y}(r)$.
3) $\overline{(kx)}(r)=k\bar{x}(r)$.
4) $\underline{(kx)}(r)=k\underline{x}(r)$.

Definition 2.2:
We say that a fuzzy number vector $(x_1,x_2,....,x_n)^t$ given by $x_i=(\underline{x^i}(r),\bar{x_i}(r))$, $1\leqslant i\leqslant n$, $0\leqslant r\leqslant 1$, is called a solution of the FSLE if

$$\underline{\sum_{j=1}^n a_{ij}x_j}=\sum_{j=1}^n\underline{a_{ij}x_j}=\underline{y_i},$$

$$\overline{\sum_{j=1}^n a_{ij}x_j}=\sum_{j=1}^n\bar{a_{ij}x_j}=\bar{y_i},\tag{6}$$

Either the ith equation of the system (5):
$a_{i1}(\underline{x_1},\bar{x_1})+...+a_{ii}(\underline{x_i},\bar{x_i})+...+a_{in}(\underline{x_n},\bar{x_n})=(\underline{y_i},\bar{y_i})$,
we have

$$\begin{aligned}\underline{a_{i1}x_1}+...+\underline{a_{ii}x_i}+...+\underline{a_{in}x_n}&=\underline{y_i}(r)\\ \bar{a}_{i1}x_1+...+\bar{a}_{ii}x_i+...+\bar{a}_{in}x_n&=\bar{y}_i(r),\\ 1\leqslant i\leqslant n,0\leqslant r&\leqslant 1.\end{aligned}\tag{7}$$

as what you see from (8) for any i we have two linear systems that there can be prolonged to a $2n\times 2n$ crisp linear system as follows :

$$SX=Y\tag{8}$$

$$\rightarrow\begin{bmatrix}T\geqslant 0 & H\geqslant 0\\ H\geqslant 0 & T\geqslant 0\end{bmatrix}\begin{bmatrix}\underline{X}\\ -\bar{X}\end{bmatrix}=\begin{bmatrix}\underline{Y}\\ -\bar{Y}\end{bmatrix}.$$

where the nonnegative entries of A are contained in T, and the absolute values of the non posetif entries are contained in H and $A=T-H$ furthermore assume that $T=L_1+D_1+U_1$. Thus we have $S=L+D+U$, where:

L=$\begin{bmatrix}L_1 & 0\\ H & L_1\end{bmatrix}$, D=$\begin{bmatrix}D_1 & 0\\ 0 & D_1\end{bmatrix}$, U=$\begin{bmatrix}U_1 & H\\ 0 & U_1\end{bmatrix}$

We now notice that the system of linear equation (9) is a $2n\times 2n$ crisp linear system,then we can say that this system can be solved in a unique way for X if and only if the matrix S is invertible.
Yet that even though if the original matrix A is not singular, S may be.
The next results point out the difficulties for getting the fuzzy solution for a linear system.

Theorem 2.3: If the matrices $A=T-H$ and $T+H$ are invertible, then The matrix S is also invertible.

Definition 2.4:
we note by X=$\{(\underline{x_i}(r),\bar{x_i}(r)),1\leqslant i\leqslant n\}$ the unique solution of SX = Y.

Let the fuzzy number vector U= $\{(\underline{u_i}(r), \bar{u}_i(r)), 1 \leqslant i \leqslant n\}$ denote by

$$\underline{u_i}(r) = min\{\underline{x_i}(r), \bar{x}_i(r), \underline{x_i}(1)\} \quad (9)$$
$$, \bar{u}_i(r) = max\{\underline{x_i}(r), \bar{x}_i(r), \underline{x_i}(1)\},$$

Theorem 2.5:
Let S be invertible, if S^{-1} is nonnegative, then the unique solution X of equation (9) is a fuzzy vector for arbitrary vector Y continually.

III. The first-order Richardson iterative method

A. The stationary Richardson method

Consider the following linear system

$$Ax = b \quad (10)$$

The stationary first-order Richardson's method is a simplest iterative method together with a local parameter α for the speed of iteration process. Its scheme is written like this:

$$x^{(k+1)} = x^{(k)} + \alpha r^{(k)}, \quad k \geq 0 \quad (11)$$

Here $r^{(k)}$ is the residual vector of the current iterate :

$$r^{(k)} := b - x^{(k)}$$

The following results are inspired from [8] and [10].

Theorem 3.1:
The stationary Richardson scheme is convergent iff $\frac{2Re(\lambda_i)}{\alpha|\lambda_i|^2} \succ 1$, for all i=1,...,n, and $\lambda_i \in \mathbb{C}$ denotes the eigenvalues of the matrix A.

Theorem 3.2: Assume that, A has non negative real eigenvalues, orderly in a following way $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \succ 0$. Then, the stationary Richardson method (12) is convergent iff $0 \prec \alpha \prec \frac{2}{\lambda_1}$. furthermore, the optimal value of α is $\alpha_{opt} = \frac{2}{\lambda_1+\lambda_2}$.

Corollary 3.3: In order that matrix A is symmetric positive definite. Then the stationary Richardson method is convergent.

Remark 1: Despite of its convergence, the stationary Richardson's method has the inconvenience of being numerically unsteady.

B. The nonstationary Richardson method

More generally, permitting α in (12) according to on the iteration index, the nonstationary Richardson method or semi-iterative method is defined by

$$x^{(k+1)} = x^{(k)} + \alpha_k r^{(k)}, \quad k \geq 0 \quad (12)$$

The euclidean space $\mathbb{R}^n$ is equipped by the canonical scalar product <.,.> and ||.|| denotes its corresponding norm.
Since $r^{(k+1)} = r^{(k)} - \alpha A r^{(k)}$, it follows

$$||r^{(k+1)}||^2 = ||r^{(k)}||^2 - 2\alpha < r^{(k)}, Ar^{(k)} > + \alpha^2 ||Ar^{(k)}||^2$$

The optimal acceleration parameter α can be dynamically computed at every step k by

$$\frac{d}{d\alpha}||r^{(k+1)}||^2 = 0$$

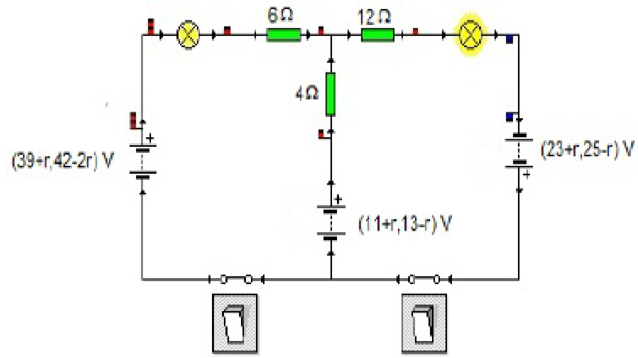

Fig. 2: A electrical circuit with fuzzy current and fuzzy source

We obtain the minimizer parameter by the next formula

$$\alpha_{opt} = \frac{< r^{(k)}, Ar^{(k)} >}{||Ar^{(k)}||^2} \quad (13)$$

Remark 2: The nonstationary Richardson method using (14) to rating the acceleration parameter, is equally named the gradient method with dynamic parameter.

Theorem 3.4: Let A be a symmetric and positive definite matrix; then the nonstationary Richardson method is convergent for each choice of the initial data $x^{(0)}$.

IV. Numerical example in Circuit Analysis

Consider a straightforward resistive circuit with fuzzy current and a source fuzzified.

This circuit has a system of equations given by:

$$10\tilde{I}_1 - 4\tilde{I}_2 = (39 + r, 42 - 2r) - (11 + r, 13 - r)$$
$$-4\tilde{I}_1 + 16\tilde{I}_2 = (11 + r, 13 - r) + (23 + r, 25 - r). \quad (14)$$

We can simplify system as:

$$\begin{cases} 10\tilde{I}_1 - 4\tilde{I}_2 = (26 + 2r, 31 - 3r) \\ -4\tilde{I}_1 + 16\tilde{I}_2 = (34 + 2r, 38 - 2r) \end{cases}$$

Therefore $S_1 = \begin{bmatrix} 10 & 0 \\ 0 & 16 \end{bmatrix}$ $S_2 = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$

and then $S = \begin{bmatrix} 10 & 0 & 0 & -4 \\ 0 & 16 & -4 & 0 \\ 0 & -4 & 10 & 0 \\ -4 & 0 & 0 & 16 \end{bmatrix}$, $\tilde{Y} = \begin{bmatrix} 26 + 2r \\ 34 + 2r \\ 31 - 3r \\ 38 - 2r \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & 0 & \frac{1}{36} \\ 0 & \frac{5}{72} & \frac{1}{36} & 0 \\ 0 & \frac{1}{36} & \frac{1}{9} & 0 \\ \frac{1}{36} & 0 & 0 & \frac{5}{72} \end{bmatrix} \quad \tilde{I} = \begin{bmatrix} \frac{71}{18} + \frac{1}{6}r \\ \frac{29}{9} + \frac{1}{18}r \\ \frac{79}{18} - \frac{5}{18}r \\ \frac{121}{36} - \frac{1}{12}r \end{bmatrix}$$

$$\tilde{I}_1 = (\tfrac{71}{18} + \tfrac{1}{6}r, \tfrac{79}{18} - \tfrac{5}{18}r), \ \tilde{I}_2 = (\tfrac{29}{9} + \tfrac{1}{18}r, \tfrac{121}{36} - \tfrac{1}{12}r)$$

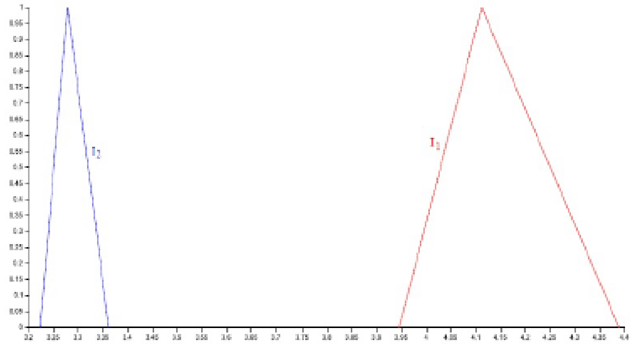

Fig. 3: The exact solution of the system in example

In our case, the matrix A is symmetric definite positive with eigenvalues 8 and 18. Next, the optimal of the minimizer parameter is $\frac{2}{18+8}$

Using the Richardson successive recursion, we obtain the approximate solution given by

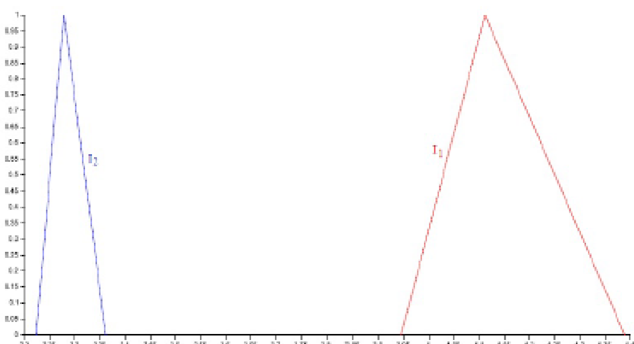

Fig. 4: The approximate solution of the system in example

V. Conclusion

In this this work, we presented the first-order Richardson approach applied to crisp systems of linear equations. Next, we adapted this method to fuzzy systems of linear equations. Finally, a pratical example of circuit analysis is given to highlight the utility of iterative methods in the fuzzy framework.

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Focus on the Method applied to the research on project management in Morocco

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Abstract—Several interesting studies have been carried out on project management and have given rise to important indicators that allow analysis, conclusions and recommendations. My ambition is to do a similar work but focused only on projects in Morocco since this work has never been done in a detailed and precise way. The final objective of this study is to:
- reflect the state of project management as performed on Moroccan companies;
- give companies the opportunity to optimize the control of their small and large projects by being inspired by the meaningful and interesting experience of the various projects carried out in Morocco.
In this paper, I will begin by explaining how I chose to conduct my study in order to extract the actual data, the method of analysis I opted for, and then focus on the target sample and its behaviour.

Keywords—Optimization, project management, failure, success, statistics, method, study.

I. Introduction

It was possible to concentrate my study on a single company with several types of projects but this option was without any interest for me for several reasons:

- In my professional career I have already had the opportunity to participate and manage projects of different sizes in several companies. So I already know how it works in detail.

- A single company or even two or three remain insufficient case studies for a generalization.

- Companies prefer to keep their project failures hidden and refuse to talk about them, especially if the objective is a publication.

- The objective of the study is to know the trends of project management in Morocco. Each project is unique and its progress cannot be generalized to other projects. This is why it is important for me to study the experience of several projects in several sectors and fields of activity and to provide a vision from several angles and by different profiles.

The way in which I proceeded in order to have real data from the field is based on two study methods. Firstly, interviews with different executives in different fields and sectors followed by an online survey in anonymous mode in order to respect the confidentiality of the companies [1][2].

II. Method of the Study

A. *Interviews*

The target of about twenty interviews was composed of 50% of executives working in IT departments and organizations of large structures in Morocco. The remaining 50% are either employees in the public sector, engineers in small and medium enterprises, or people who have created their own small business. The objective was to have a maximum of information from different sources in order to prepare a work that will concern a larger population.

B. *Survey*

The survey has been activated during the whole month of December 2020. It was composed by a series of 27 questions and was addressed to different people who have participated or managed projects in companies of all sizes in both the private and public sectors.

Initially the survey was destined only for people whose job was the IT project, consulting firms, IT services companies and information systems departments of companies of different activities. But then I thought it would be interesting to involve even people who have jobs other than project management but who have been involved in projects. The total population targeted was over 400 people, and came up with 256 complete, serious and significant responses.

An invitation to participate in the survey was sent to contacts, friends and ex-work colleagues who had already had project experience. Then another invitation concerned my old colleagues at school, computer engineers, who almost all of them work in the field of project management. A request was addressed to them to transfer the same survey to their collaborators and colleagues at work. Finally, the survey was published in groups specialized in project management.

III. Interview Results

It appears from the interviews that:

- Project management is a real issue that concerns and preoccupies managers in all areas and sectors [3][4].
- Most individuals who have carried out personal projects do not follow any project method. Everything is managed on a day-to-day basis without any risk calculation or planning. This applies to even the largest personal projects. This practice often impacts the time and quality parameters. The focus of the project owner is more on the cost component which is his first concern. This is why we note that in Morocco most personal and family projects take a long time to be realized, going through several problems that could have been avoided by a simple risk management, and the delivery is often of average quality since no quality control is imposed.
- Engineers in IT departments and organizations are the most affected and aware of project management issues at all levels. Difficulties to manage project teams, to manage customers who generally have no limits in their requests, to have the means they need to carry out the project, to have the support and attention of top management...
- The trend of most engineers in large structures is towards the integration of agile in their management, which is giving very good results so far. They find many difficulties in the application of this method since it is new and being experimented in Morocco [6-9].
- The companies do not organize training for their employees who train themselves in their own way and according to their means. This is a risk not to be neglected. We will see later in this study the verification of this point during the survey.

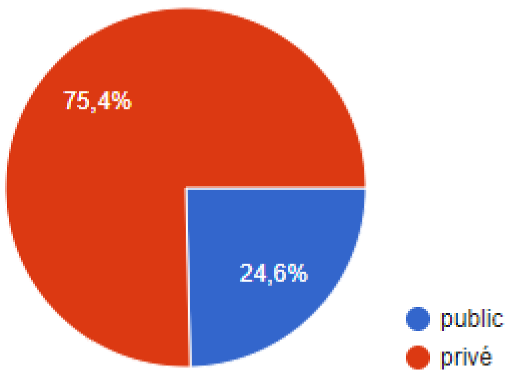
IV. Survey Analysis Approach

The analysis of the survey will be an exploratory descriptive analysis of the data. It will allow the observed data to be represented in the simplest and easiest form possible and easy to read and analyze. These representations will allow us to carry out several analyses. First, analyze the result of each question separately and then cross the questions to obtain more focused and precise conclusions.

This study is the first targeted and complete survey of its kind on project management in Morocco with the objective of optimizing this practice. Therefore, I cannot use any previous knowledge to compare the results of the study in a controlled manner. The survey is essentially exploratory, and tries to identify trends in project management in Morocco through the analysis of a targeted population that the selected sample will represent.

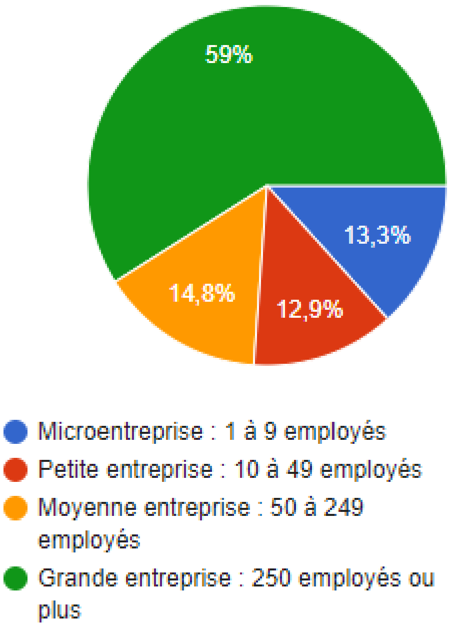
V. Focus on the Sample

“Fig. 1” shows the distribution of firms in the sample by private/public sector. We note that the private sector is over-represented compared to the public sector. This has been planned from the beginning, which is quite logical since the project management business clearly concerns more private companies that necessarily look to optimize their performance while reducing costs and delays, the main reason why project management methods are adopted. But it is important to have a vision of project management in the public sector as well, which will be the case in my study.



“Fig. 1”

Companies of all sizes have been focused to have a consistent and complete representation. But the majority of the population works in large companies with more than 250 employees, why? Because in large companies the number and size of projects is greater and more important, which necessarily requires the application of a good methodology to manage them and make them successful [5][6]. But we also have the micro, small and medium enterprise vision which will allow us to have a complete view.



"Fig. 2"

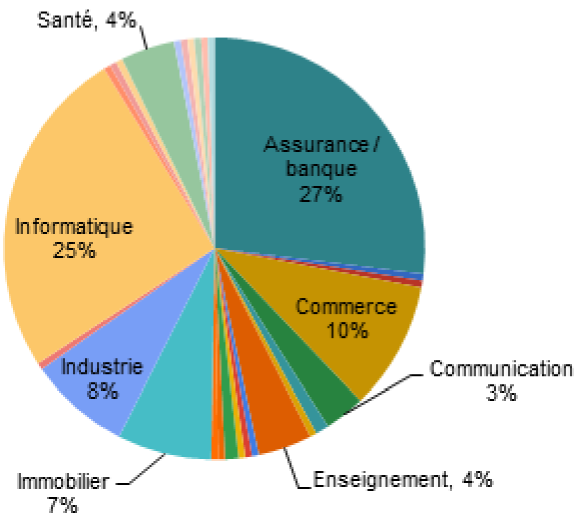
As you can see in *"Fig. 3"* below, all trades have been targeted and approached to see how all types of projects and in all sectors. It is clear that the areas most concerned by project management and which appeared the most in our sample are:

- IT and banking/insurance (project management) in 1st position,

- business, building and industry in 2nd position,

- health, communication and education in 3rd position,

- then come all the other trades.

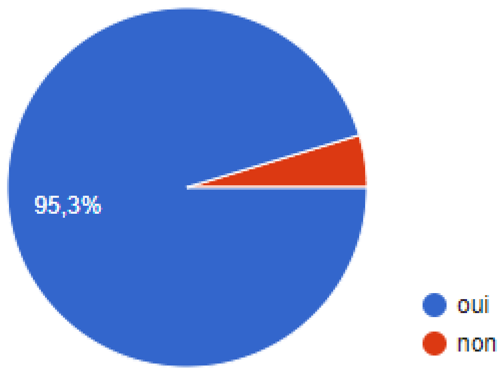


"Fig. 3"

This was in accordance with my expectations and the definition of my sample set before the study was launched.

This will allow us to study project management domain by domain in order to highlight problems and difficulties affecting the success of projects.

I asked if the company in which you work is aware of the importance of adopting a method in the management of its projects in order to improve their management and success. The answer was both surprising and satisfying at the same time. Almost 100% responded positively as shown in "Fig.4" below.

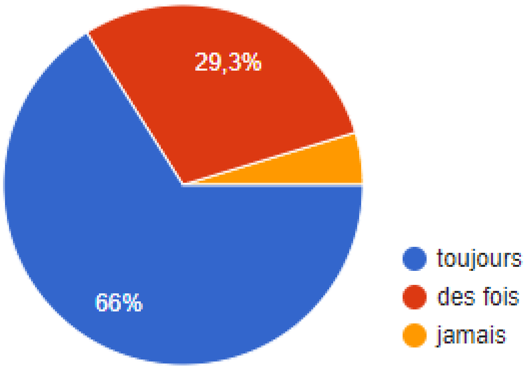


"Fig.4"

Analyzing the previous answer we can think that everyone adopts a project management method to achieve their objectives since they are aware of it. But the answer is not what I expected.

Already we can see that those who answered not being aware of the importance of this practice are exactly those who never apply it. Which is quite normal and logical, they represent 4.7% of our sample, which is insignificant.

66% of the sample always manages its projects according to a management method; the remaining 29.3% do not apply it all the time. This is a good rate, although companies here in Morocco are mostly aware of the importance of project management but they do not all apply a method necessarily all the time. This can be explained by the lack of skills maybe, lack of training, lack of resources ... we will have the opportunity to see in the study the real reason behind [10][11].



"Fig. 5"

VI. Conclusion

This was a presentation of the methodology and approach I followed for my project management optimization study in Morocco where I was able to explain why my choice was made this way and not in any other way. A zoom on the studied population was also presented. The study will continue in order to move on to a detailed analysis of the results of the survey carried out according to the method explained above.

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