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Optimality Conditions for Interval-Valued Optimization Problems: A Real-Valued Optimization Approach

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***Abstract*—This paper explores a class of interval-valued optimization problems (IVOPs). An associated scalar optimization problem (ASOP) and a constrained optimization problem (ACOP) are formulated associated with IVOP. It is demonstrated that any optimal solution of either ASOP or ACOP qualifies as an LU-solution of IVOP. Additionally, sufficient optimality conditions for IVOP are derived under strong convexity assumptions on the objective function. To illustrate the practical significance of these conditions, they are applied to solving complex IVOPs using MATLAB R2024a.**

***Index Terms*—Interval-valued optimization, Optimality conditions, Generalized Hukuhara derivative, Constraint optimization problems**

I. INTRODUCTION

It has been observed that many optimization methods existing in the literature assume precise values for objective functions and constraints (see, [1], [9]). However, real-world situations often involve inherent uncertainties in data or model parameters, rendering classical optimization techniques inadequate for such models (see, [4], [18]). Consequently, various probabilistic and non-probabilistic approaches, such as stochastic programming [3] and robust optimization [2], have been developed to address these uncertainties. However, these methods have their drawbacks. In stochastic programming, uncertain coefficients are treated as random variables with known distribution functions, which can be challenging to determine in critical scenarios. On the other hand, robust optimization always considers the worst-case scenario, potentially leading to overly conservative and costly solutions when favorable conditions arise.

Recently, interval analysis has emerged as a non-probabilistic approach to address imprecise information in models, where the lower and upper bounds of uncertain parameters are estimated from historical data. Building on this foundation, interval-valued optimization problems (IVOPs) provide a robust framework for managing such uncertainties by representing objectives and constraints with intervals rather than fixed values. This approach significantly enhances the applicability of IVOPs in practical optimization scenarios such as engineering and management sciences (see, [14], [16]). The initial development and formulation of IVOPs are credited to Ishibuchi and Tanaka [6], where the authors studied an IVOP by formulating a corresponding multiobjective optimization problem. Wu [15] presented sufficient optimality conditions for IVOP under Hukuhara differentiability and convexity assumptions. Further, Zhang et al. [17] employed the concept of invexity to establish sufficient optimality conditions for IVOP involving differentiable functions. Rahman et al. [11] explored necessary and sufficient optimality conditions for IVOPs, involving differentiable lower-bound and upper-bound functions of the objective function of IVOP. On the other hand, Osuna-Gómez et al. [10] established necessary optimality conditions for IVOPs involving generalized Hukuhara differentiable objective function, which is defined on a subset of $\mathbb{R}$.

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It is worthwhile to note that optimality conditions for IVOPs have been investigated by many researchers by employing the Hukuhara derivatives (see, Wu [15]) and differentiability of upper-bound and lower-bound functions of interval-valued function (see, Rahman et al. [11]). However, optimality conditions for IVOP using the notions of gH-differentiability have only been investigated when the domain of the objective function is a subset of real numbers (see, Osuna-Gómez et al. [10]). In this article, the primary objective is to introduce sufficient optimality conditions for IVOP in terms of ACOP and ASOP, enabling the use of well-developed constrained and unconstrained optimization methods for real-valued functions to solve IVOP. Necessary optimality conditions for IVOP are established by leveraging gH-differentiability. Additionally, sufficient optimality conditions are derived by imposing an additional assumption of strong convexity on the objective function of IVOP.

Motivated by the works of [10], [11], [15], this article studied a class of IVOP in the context of Euclidean spaces. For the proposed IVOP, associated scalar-valued optimization problems (ASOP) and constrained optimization problems (ACOP) are illustrated. It is shown that the optimal solutions of either ASOP or ACOP correspond to the LU-solution of IVOP. Moreover, necessary optimality conditions for the existence of the LU-solution of IVOP are established under generalized partial Hukuhara differentiability assumptions on the objective function. Sufficient optimality conditions for IVOP are also derived under strong convexity hypotheses. For illustrative purposes, the results are complemented with numerical examples.

The novelty and contributions of this article are discussed as follows. The approach of converting IVOP to ASOP and ACOP enables us to solve IVOP by utilizing well-established numerical techniques designed for addressing both constrained and unconstrained optimization problems of real-valued functions. Furthermore, necessary and sufficient optimality conditions for IVOP are established under appropriate assumptions. The results presented in this paper extend and generalize several significant findings already existing in the literature. Specifically, the optimality conditions for IVOP are real numbers established by Osuna-Gómez et al. [10] has been extended to the optimality condition for IVOP on $\mathbb{R}^n$.

The structure of the paper is outlined as follows. Section II provides a review of fundamental definitions, notations, and terminologies that are essential for subsequent discussions. Section III focuses on establishing the optimality conditions for IVOP and is divided into two parts. In Subsection III-A, ASOP and ACOP corresponding to IVOP are introduced, and the relationship between the solutions of ACOP and the effective solutions of IVOP is explored. To illustrate the practical applicability of these results, several non-trivial numerical examples are presented. In Subsection III-B, the necessary optimality conditions for IVOP are established using gH-differentiability, and sufficient optimality conditions are derived under the additional assumption of strong convexity of the objective function. Finally, Section V concludes the paper by summarizing the main findings and suggesting potential directions for future research.

II. Notations and Mathematical Preliminaries

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The symbol $\mathbb{R}^k$ and $\mathbb{N}$ will be used to denote the k-dimensional Euclidean space and the set of all natural numbers, respectively. For $l \in \mathbb{N}$, the set $\{1, 2, \ldots, l\}$ will be denoted by $\mathcal{E}_l$. For any set $\mathcal{G}$, the symbol $\mathcal{G}^k$ will be defined as follows:

$$\mathcal{G}^k := \underbrace{\mathcal{G} \times \cdots \times \mathcal{G}}_{k \text{ times}}.$$

Unless explicitly mentioned, the symbols $\mathcal{Z}$ and $\mathcal{S}$ will be used to denote non-empty open subsets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively. The following definition is from [**?**].

Definition II.1 Let

$$\Upsilon$$

:S $\to \mathbb{R}$ be a real-valued function and $\hat{a} \in \mathcal{S}$. Then, the lateral derivatives of T at the point $\hat{a}$ are defined as follows:

$$\Upsilon'_+(\hat{a}) := \lim_{\beta \to 0^+} \frac{\Upsilon(\hat{a} + \beta) - \Upsilon(\hat{a})}{\beta},$$

and $\Upsilon'_-(\hat{a}) := \lim_{\beta \to 0^-} \frac{\Upsilon(\hat{a}+\beta)-\Upsilon(\hat{a})}{\beta}$, provided both the limits exist.

The set $\{[\underline{u}, \overline{u}] : \underline{u}, \overline{u} \in \mathbb{R} \text{ and } \underline{u} \leq \overline{u}\}$ referred to as $\mathcal{I}(\mathbb{R})$. The collection of all the elements $[\underline{u}, \overline{u}] \in \mathcal{I}(\mathbb{R})$, satisfying $\underline{u} \leq \overline{u}$ is denoted by $\mathcal{I}(\mathbb{R})$. For $\xi_1, \xi_2 \in \mathbb{R}$, the symbol $[\xi_1 \vee \xi_2]$ is employed to denote the interval $[\min\{\xi_1, \xi_2\}, \max\{\xi_1, \xi_2\}]$. For any real number u, the interval $[u, u]$ is referred to as a degenerate interval, and vice versa.

Let $\mathcal{U} = (u_1, u_2, \ldots, u_k) \in \mathbb{R}^k$, then we say that

$$\mathcal{U} \in [\underline{z}, \overline{z}] := [z_1, z_2, \ldots, z_k] \in \mathcal{I}(\mathbb{R})^k$$

if and only if $u_j \in [\underline{z}_j, \overline{z}_j]$ for every $j \in \mathcal{E}_k$.

The interval-valued function $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as:

$$\Upsilon(z) := [\underline{\Upsilon}(z), \overline{\Upsilon}(z)], \qquad \text{for every } z \in \mathcal{Z},$$

where $\mathcal{Z}$ is a non-empty subset of $\mathbb{R}^n$. The functions $\underline{\Upsilon}$ and $\overline{\Upsilon}$ are real-valued functions referred to as lower-bound and upper-bound functions of Υ, respectively.

Let $\mathcal{W} := [\underline{w}, \overline{w}]$, $\mathcal{X} := [\underline{x}, \overline{x}] \in \mathcal{I}(\mathbb{R})$ be arbitrary. The following algebraic operations from [**?**] will be used in the sequel:

$$\mathcal{W} \oplus \mathcal{X} := \{w + x : w \in \mathcal{W},\, x \in \mathcal{X}\} = [\underline{w} + \underline{x},\, \overline{w} + \overline{x}],$$

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$\mathcal{W} \ominus \mathcal{X} := \{w - x : w \in \mathcal{W},\, x \in \mathcal{X}\} = [\underline{w} - \overline{x},\, \overline{w} - \underline{x}]$.

$$\text{k} \odot \mathcal{W} := \{kw : w \in \mathcal{W}\} = \begin{cases} [k\underline{w},\, k\overline{w}], & k \geq 0, \\ [k\overline{w},\, k\underline{w}], & k \leq 0. \end{cases}$$

We recall the following ordered relations from [?]. Let $\mathcal{W} := [\underline{w}, \overline{w}]$, $\mathcal{X} := [\underline{x}, \overline{x}] \in \mathcal{I}(\mathbb{R})$. The ordered relations between $\mathcal{W}$ and $\mathcal{X}$ are defined as follows:

$$\mathcal{W} \leq_{LU} \mathcal{X} \iff \underline{w} \leq \underline{x} \text{ and } \overline{w} \leq \overline{x},$$

$$\mathcal{W} <_{LU} \mathcal{X} \iff \mathcal{W} \leq_{LU} \mathcal{X} \text{ and } \mathcal{W} \neq \mathcal{X},$$

$$\mathcal{W} <_{LU} \mathcal{X} \iff \underline{w} < \underline{x} \text{ and } \overline{w} < \overline{x}.$$

Let $\mathcal{W}^k = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k)$, $\mathcal{X}^k = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k) \in (\mathcal{I}(\mathbb{R}))^k$, then the ordered relations between $\mathcal{W}$ and $\mathcal{X}$ are defined as follows:

$$\mathcal{W}^k \leq_k \mathcal{X}^k \iff \mathcal{W}_i \leq_{LU} \mathcal{X}_i, \quad \text{for every } i \in \mathcal{E}_k,$$

$$\mathcal{W}^k <_k \mathcal{X}^k \iff \mathcal{W}_i <_{LU} \mathcal{X}_i, \quad \text{for every } i \in \mathcal{E}_k.$$

The following definitions are from Wu [?].

Definition II.2 The gH-difference between $\mathcal{W} := [\underline{w}, \overline{w}]$ and $\mathcal{X} := [\underline{x}, \overline{x}] \in \mathcal{I}(\mathbb{R})$, denoted by $\mathcal{W} \ominus_{gH} \mathcal{X}$, is defined as follows:

$$\mathcal{W} \ominus_{gH} \mathcal{X} := [\min\{\underline{w} - \underline{x},\, \overline{w} - \overline{x}\},\, \max\{\underline{w} - \underline{x},\, \overline{w} - \overline{x}\}].$$

Definition II.3 Let $\mathcal{Z} \subseteq \mathbb{R}^n$ be a convex set. The function $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is termed as a convex function if, for any $z_1, z_2 \in \mathcal{Z}$ and $\beta \in [0, 1]$, the following inequality is satisfied:

$$\Upsilon((1 - \beta)z_1 + \beta z_2) \leq_{LU} (1 - \beta) \odot \Upsilon(z_1) \oplus \beta \odot \Upsilon(z_2).$$

The following definition revisits the concept of strongly convex interval-valued functions.

Definition II.4 Let $\mathcal{Z} \subseteq \mathbb{R}^n$ be convex. The function $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is said to be strongly convex on $\mathcal{Z}$, if there exists $\gamma > 0$
such that for all $z_1 \neq z_2 \in \mathcal{Z}$ and $\beta \in (0, 1)$, the following inequality is satisfied:

$$\begin{aligned} &\Upsilon\big((1 - \beta)z_1 + \beta z_2\big) \oplus \beta \|z_2 - z_1\|^2 \odot [0, \gamma] \\ &\quad \leq_{LU} (1 - \beta) \odot \Upsilon(z_1) \oplus \beta \odot \Upsilon(z_2). \end{aligned}$$

We recall the following definition and theorem from Stefanini and Bede [?].

Definition II.5 Let $\hat{a} \in \mathcal{S}$ and let $\Upsilon : \mathcal{S} \to \mathcal{I}(\mathbb{R})$ be an interval-valued function. The gH-derivative of Υ at $\hat{a}$, denoted by $\Upsilon'_{gH}(\hat{a})$, is defined as follows:

$$\Upsilon'_{gH}(\hat{a}) := \lim_{\beta \to 0} \frac{\Upsilon(\hat{a} + \beta) \ominus_{gH} \Upsilon(\hat{a})}{\beta},$$

provided the limit exists.

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$$\Upsilon'_{gH}(\hat{a}) := \lim_{\beta \to 0} \frac{\Upsilon(\hat{a} + \beta) \ominus_{gH} \Upsilon(\hat{a})}{\beta},$$

provided the limit exists.

Theorem II.1 Let $\Upsilon : \mathcal{S} \to \mathcal{I}(\mathbb{R})$ be an interval-valued function defined by:

$$\Upsilon(z) := [\underline{\Upsilon}(z),\, \overline{\Upsilon}(z)], \qquad \text{for every } z \in \mathcal{S}.$$

The function Υ is gH-differentiable at $\hat{a} \in \mathcal{S}$ if and only if one of the following holds:

(a)

1) $\underline{\Upsilon}$ and $\overline{\Upsilon}$ are differentiable at $\hat{a}$ and

$$\Upsilon'_{gH}(\hat{a}) = \left[\min\{\underline{\Upsilon}'(\hat{a}),\, \overline{\Upsilon}'(\hat{a})\}, \max\{\underline{\Upsilon}'(\hat{a}),\, \overline{\Upsilon}'(\hat{a})\}\right].$$

2) The left and right hand derivatives of $\underline{\Upsilon}$ and $\overline{\Upsilon}$ at $\hat{a}$, that is, $\underline{\Upsilon}'_-(\hat{a})$, $\underline{\Upsilon}'_+(\hat{a})$, $\overline{\Upsilon}'_-(\hat{a})$, $\overline{\Upsilon}'_+(\hat{a})$, exist and satisfy

$$\underline{\Upsilon}'_-(\hat{a}) = \overline{\Upsilon}'_-(\hat{a}), \qquad \underline{\Upsilon}'_+(\hat{a}) = \overline{\Upsilon}'_+(\hat{a}).$$

Moreover,

$$\Upsilon'_{gH}(\hat{a}) = \left[\min\{\underline{\Upsilon}'_+(\hat{a}),\, \overline{\Upsilon}'_+(\hat{a})\}, \max\{\underline{\Upsilon}'_+(\hat{a}),\, \overline{\Upsilon}'_+(\hat{a})\}\right].$$

The following definitions are from Stefanini Arana-Jiménez [?]. **Definition II.6** Let $\hat{a} := (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \in \mathcal{Z}$ and let $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ be an interval-valued function. For a fixed $i \in \mathcal{E}_n$, the i^{th} partial generalized Hukuhara derivative (abbreviated as partial gH-derivative), denoted by $\frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_i}$, is defined as follows:

$$\frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_i} := \lim_{\beta \to 0} \frac{\Upsilon(\hat{a}_1, \ldots, \hat{a}_i + \beta, \ldots, \hat{a}_n) \ominus_{gH} \Upsilon(\hat{a}_1, \ldots, \hat{a}_i, \ldots, \hat{a}_n)}{\beta}$$

provided the limit exists.

If $\frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_i}$ exists for all $i \in \mathcal{E}_n$, then the gH-gradient of Υ at $\hat{a}$ is defined as follows:

$$\nabla_{gH}\Upsilon(\hat{a}) := \left(\frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_1}, \frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_2}, \ldots, \frac{\partial_{gH}\Upsilon(\hat{a})}{\partial \hat{a}_n}\right).$$

Definition II.7 Let $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ be an interval-valued function and $\hat{a} \in \mathcal{Z}$. The gH-directional derivative of Υ in the direction $d \in \mathbb{R}^n$, denoted by $D_{gH}\Upsilon(\hat{a}, d)$, is defined as follows:

$$D_{gH}\Upsilon(\hat{a}, d) := \lim_{h \to 0} \frac{\Upsilon(\hat{a} + hd) \ominus_{gH} \Upsilon(\hat{a})}{h},$$

provided the limit exists.

We now consider the following scalar optimization problem:

$$\text{(SOP)} \qquad \text{Minimize} \quad \Lambda(z),$$

$$\text{subject to} \quad z \in \mathcal{Z},$$

where $\Lambda : \mathcal{Z} \to \mathbb{R}$ represents a real-valued function. In the following definition, we recall the notion of a minimizer for the problem SOP (see, Mishra and Upadhyay [**?**]).

Definition II.8 An element $\hat{z} \in \mathcal{Z}$ is termed as a global minimizer of the problem SOP, if the following inequality is satisfied:

$$\Lambda(\hat{z}) \leq \Lambda(z), \quad \text{for every } z \in \mathcal{Z}. \tag{II.1}$$

Furthermore, if for every $z \in \mathcal{Z}$ and $z \neq \hat{z}$, the following holds:

$$\Lambda(\hat{z}) < \Lambda(z), \tag{II.2}$$

then $\hat{z}$ is called strict global minimizer of SOP. If there exists a non-empty open subset $\mathcal{N}$ of $\mathbb{R}^n$, such that (II.1) and (II.2) are satisfied for all $z \in \mathcal{Z} \cap \mathcal{N}$, then the point $\hat{z}$ is referred to as a local minimizer and a strict local minimizer of SOP, respectively.

IV. Optimality Conditions for IVOP

This section delves into the optimality conditions for IVOP and is divided into two subsections. In the first subsection, sufficient optimality conditions for IVOP are developed in terms of constrained and unconstrained real-valued optimization problems. This formulation facilitates the use of well-established optimization techniques for real-valued functions, making the solution of IVOP more effective and practical. In the second subsection, necessary and sufficient optimality conditions are established by employing the gH-derivative. This approach extends the existing necessary optimality conditions found in the literature, offering a more comprehensive framework for tackling IVOP. Together, these methodologies aim to bridge the theoretical and practical gap between interval-valued and real-valued optimization techniques, enhancing both the theoretical understanding and the practical applicability of IVOP solutions.

The interval-valued optimization problem under consideration is formulated as follows:

$$\text{(IVOP)} \qquad \text{Minimize} \quad \Upsilon(z),$$

$$\text{subject to} \quad z \in \mathcal{Z},$$

where $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as:

$$\Upsilon(z) := [\underline{\Upsilon}(z), \overline{\Upsilon}(z)], \qquad \text{for every } z \in \mathcal{Z},$$

where $\underline{\Upsilon}$ and $\overline{\Upsilon}$ defined on $\mathcal{Z}$ are referred to as lower-bound and upper-bound functions of Υ, respectively. The following definition of LU-solution of (IVOP) is from Ishibuchi, H, and Tanaka [**?**].

Definition III.1 An element $\hat{z} \in \mathcal{Z}$ is said to be a global LU-solution of IVOP if the following inequality is not satisfied for any $z \in \mathcal{Z}$:

$$\Upsilon(z) \leq_{LU} \Upsilon(\hat{z}). \tag{III.1}$$

A point $\hat{z}$ is called a local LU-solution of IVOP, if there exists a neighbourhood $\mathcal{N}$ of $\hat{z}$, such that (III.1) is not satisfied for any $z \in \mathcal{Z} \cap \mathcal{N}$.

A. Optimality conditions for IVOP: A real-valued optimization approach

In this subsection, two real-valued optimization problems associated with IVOP are introduced. Additionally, the relationship between the optimal solution of IVOP and the optimal solutions of these associated problems is established.

First, corresponding to the IVOP discussed earlier, an associated scalar-valued optimization problem (ASOP) is formulated as follows:

$$\text{(ASOP)} \quad \text{Minimize} \quad \Lambda(z,\mu) := \underline{\Upsilon}(z) + \mu\big(\overline{\Upsilon}(z) - \underline{\Upsilon}(z)\big),$$

$$\text{subject to} \quad (z,\mu) \in \mathcal{Z} \times [0,1],$$

where $\Lambda : \mathcal{Z} \times [0,1] \to \mathbb{R}$ is a real-valued function. The following theorem establishes a relationship between the minimizer of the ASOP and the LU-solution of the IVOP.

Theorem III.1 Let $(\hat{z}, \hat{\mu}) \in \mathcal{Z} \times [0,1]$ be a minimizer of (ASOP). If $\hat{\mu} \in (0,1)$ then $\hat{z} \in \mathcal{Z}$ is an LU-solution of the (IVOP).

Proof. From the provided hypotheses, $(\hat{z}, \hat{\mu}) \in \mathcal{Z} \times (0,1)$ is a minimizer of (ASOP). Therefore, for any $(z,\mu) \in \mathcal{Z} \times [0,1]$,

$$\Lambda(\hat{z}, \hat{\mu}) \leq \Lambda(z, \mu).$$

In particular, when $\mu = \hat{\mu}$, it follows that

$$\Lambda(\hat{z}, \hat{\mu}) \leq \Lambda(z, \hat{\mu}), \qquad \text{for every } z \in \mathcal{Z}. \tag{III.2}$$

On the contrary, let us assume that $\hat{z} \in \mathcal{Z}$ is not an LU-solution of the problem IVOP. Then there exists an element $z \in \mathcal{Z}$, for which the following inequality holds:

$$\Upsilon(z) \leq_{LU} \Upsilon(\hat{z}).$$

From the above inequality, it follows that:

$$\underline{\Upsilon}(z) \leq \underline{\Upsilon}(\hat{z}) \quad \text{and} \quad \overline{\Upsilon}(z) \leq \overline{\Upsilon}(\hat{z}),$$

or

$$\overline{\Upsilon}(z) \leq \overline{\Upsilon}(\hat{z}) \quad \text{and} \quad \underline{\Upsilon}(z) \leq \underline{\Upsilon}(\hat{z}).$$

Therefore, we arrive at the following:

$$\Lambda(z, \hat{\mu}) < \Lambda(\hat{z}, \hat{\mu}),$$

which contradicts (III.2). This completes the proof.

Corollary 1 Let $(z, \mu) \in \mathcal{Z}\times[0,1]$ is a local minimizer of the problem ASOP. If $\mu \in (0,1)$, then $\hat{z} \in \mathcal{Z}$ is a local LU-solution of the problem IVOP. *Proof.* The proof follows along similar lines as the proof of Theorem III.1.

Remark III.1 If $\hat{z}$ is a local LU-solution of IVOP, then there is no guarantee that we can always get $\hat{\mu} \in [0,1]$, such that $(\hat{z}, \hat{\mu})$ will be a local minimizer of (ASOP). This fact is illustrated in the following example.

Example III.1 Consider the following interval-valued optimization problem:

$$\text{(PIII.1.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$

$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z} := [-1,1] \times [-1,1],$$

where the function $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := [\cos(z_1+z_2),\ \sin(z_1+z_2)+2], \quad \text{for all } (z_1, z_2$$

The corresponding associated real-valued optimization problem is formulated in the following manner:

$$\text{(PIII.1.2)} \qquad \text{Minimize} \quad \Lambda((z_1,z_2),\mu) = (1-\mu)\cos(z_1+z_2)+\mu\big(\sin(z_1+z_2)+2\big),$$

$$\text{subject to} \quad ((z_1,z_2),\mu) \in \mathcal{Z} \times [0,1].$$

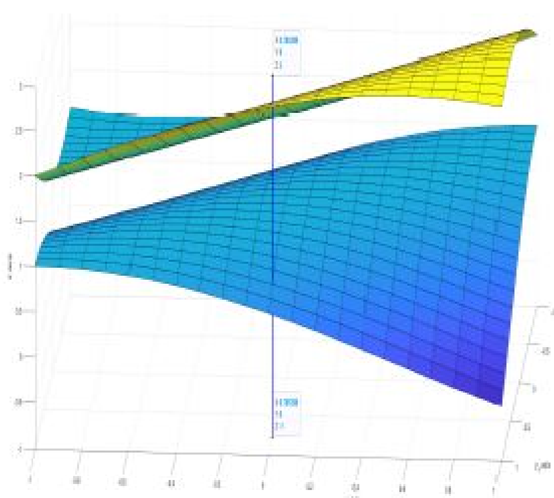

Fig. 1. $\Upsilon(x_1, x_2)$

It can be easily observed from Figure 1(a) that $\hat{z} = \left(\frac{\pi}{4}, 0\right)$ is a local LU-solution to the problem P3.1.1. However, as seen in Figure 1(b), it can be observed that $(\hat{z}, \hat{\mu})$ is not a local minimizer of P3.1.2 for any choice of $\hat{\mu} \in [0,1]$.

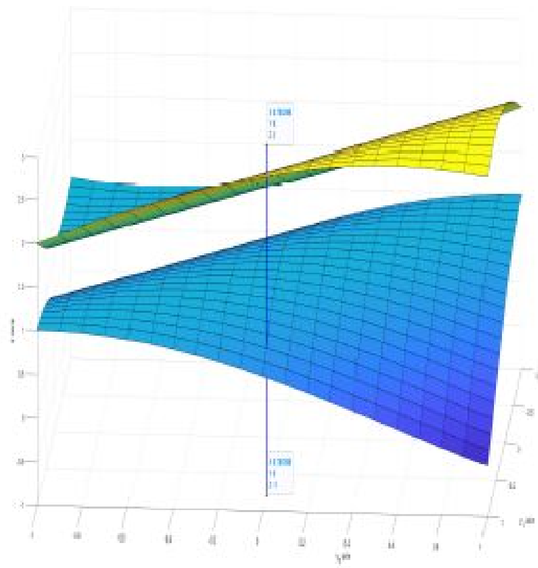

Fig. 2. $\Upsilon(x_1, x_2)$

Let $\mu \in [0,1]$ be fixed. Now, corresponding to IVOP, we formulate another associated real-valued optimization problem ASOP1 as follows:

$$\text{(ASOP1)} \qquad \text{Minimize} \quad \Lambda(z) := \underline{\Upsilon}(z)+\mu\big(\overline{\Upsilon}(z)-\underline{\Upsilon}(z)\big),$$

$$\text{subject to} \quad z \in \mathcal{Z},$$

where $\Lambda : \mathcal{Z} \to \mathbb{R}$ is a real-valued function.

Remark III.2 It is worth noting that when $\mu = 0$ or $\mu = 1$, a local minimizer of ASOP1 may not necessarily be a local LU-solution of IVOP. To illustrate this fact, we furnish the following example.

Example III.2 Consider the following interval-valued optimization problem:

$$\text{(PIII.2.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$

$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := (-1,1) \times (-1,1)$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1,z_2) := [\underline{\Upsilon}(z_1,z_2),\ \overline{\Upsilon}(z_1,z_2)] = \begin{cases} [0,\ z_1^2+z_2^2], & \text{if } z_1 \le 0, \\ [-(z_1+z_2),\ 2], & \text{if } z_1 > 0, \end{cases}$$

where $(z_1, z_2) \in \mathcal{Z}$. Further, $\underline{\Upsilon} : \mathcal{Z} \to \mathbb{R}$ and $\overline{\Upsilon} : \mathcal{Z} \to \mathbb{R}$ are defined as follows:

$$\underline{\Upsilon}(z_1,z_2) := \begin{cases} 0, & \text{if } z_1 \le 0, \\ -(z_1+z_2), & \text{if } z_1 > 0, \end{cases}$$

$$\overline{\Upsilon}(z_1, z_2) := \begin{cases} z_1^2 + z_2^2, & \text{if } z_1 \leq 0, \\ 2, & \text{if } z_1 > 0, \end{cases}$$

for every $(z_1, z_2) \in \mathcal{Z}$.

For $\mu = 0$ and $\mu = 1$, ASOP1 associated with PIII.2.1 is given by Problems PIII.2.2 and PIII.2.3, respectively, as follows:

$$\text{(PIII.2.2)} \quad \text{Minimize} \quad \Lambda(z_1, z_2) := \underline{\Upsilon}(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$
$$\text{(PIII.2.3)} \quad \text{Minimize} \quad \Lambda(z_1, z_2) := \overline{\Upsilon}(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z}.$$

The points (0.5, 0) and (0.5, 0) are the local minimizers of PIII.2.2 and PIII.2.3, respectively. However, from

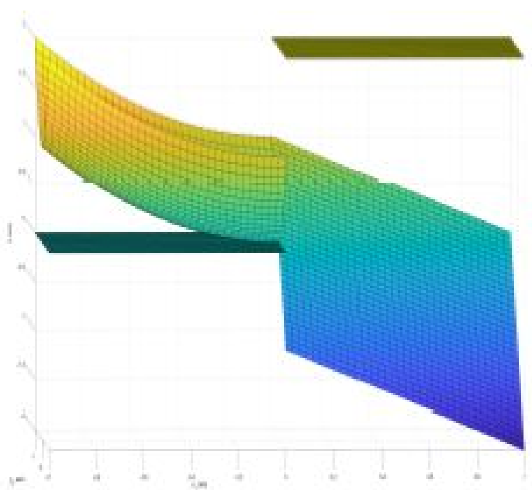

Fig. 3. $\Upsilon(x_1, x_2)$

Figure 3, it can be easily verified that these points are not local LU-solutions of PIII.2.1.

Remark III.3 When either $\mu = 0$ or $\mu = 1$ is fixed, and $\hat{z}$ is an interior point of $\mathcal{Z}$ that serves as a strict local minimizer of ACOP1, it is concluded that $\hat{z}$ is a local LU-solution of IVOP. Furthermore, if $\mu \in (0, 1)$ is fixed and $\hat{z}$ is a local minimizer of ACOP1, then $\hat{z}$ is also a local LU-solution of IVOP.

Theorem III.1 and Remark III.3 can be employed to find LU-solution of IVOP even in cases where the objective function of IVOP does not possess partial gH-derivatives. This fact is illustrated in the following example.

Example III.3 Consider the following interval-valued optimization problem:

$$\text{(PIII.3.1)} \quad \text{Minimize} \quad \Upsilon(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := (-1, 1) \times (-1, 1)$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := [\underline{\Upsilon}(z_1, z_2), \overline{\Upsilon}(z_1, z_2)] = \left[-z_1^2 - z_2^2, \ |z_1| + |z_2|\right],$$

It can be verified that

$$\lim_{h \to 0^+} \frac{\Upsilon((0,0) + h(1,0)) \ominus_{gH} \Upsilon(0,0)}{h} = [0, 1],$$

and

$$\lim_{h \to 0^-} \frac{\Upsilon((0,0) + h(1,0)) \ominus_{gH} \Upsilon(0,0)}{h} = [-1, 0].$$

Therefore, the function Υ does not possess partial gH-derivatives at the point $\hat{z} = (0, 0)$. However, since $\hat{z}$ is a strict minimizer of ASOP1 with $\mu = 1$, as per Remark III.3, $\hat{z}$ is an LU-solution of the PIII.3.1.

Second, another real-valued optimization problem related to IVOP is defined. Corresponding to the considered IVOP, the associated constrained optimization problem (ACOP) is formulated as follows:

$$\text{(ACOP)} \quad \text{Minimize} \quad t + u,$$
$$\text{subject to} \quad \overline{\Upsilon}(z) \leq t,$$
$$\underline{\Upsilon}(z) \leq u,$$
$$u \leq t.$$

where $z \in \mathcal{Z}$ and $t, u \in \mathbb{R}$. The functions $\overline{\Upsilon}, \underline{\Upsilon} : \mathcal{Z} \to \mathbb{R}$ are real-valued functions defined on the set $\mathcal{Z}$. Let the set S_Υ represent the set of feasible elements for the problem ACOP.

Definition III.2 Let $(\hat{z}, \hat{t}, \hat{u})$ be an arbitrary element of S_Υ. Then $(\hat{z}, \hat{t}, \hat{u})$ is said to be a global solution of the problem ACOP if and only if the following inequality holds for every $(z, t, u) \in S_\Upsilon$:

$$\hat{t} + \hat{u} \leq t + u. \tag{III.3}$$

Moreover, if there exists an open set $\mathcal{N} \subset \mathbb{R}^{n+2}$ such that (III.3) holds for all $(z, t, u) \in S_\Upsilon \cap \mathcal{N}$, then $(\hat{z}, \hat{t}, \hat{u})$ is said to be a local solution of ACOP.

The following theorem establishes that every solution of ACOP is also an LU-solution of IVOP.

Theorem III.2 Let $(\hat{z}, \hat{t}, \hat{u}) \in S_\Upsilon$. If $(\hat{z}, \hat{t}, \hat{u})$ is a solution to the problem ACOP, then $\hat{z}$ is an LU-solution of the problem IVOP.

Proof. On the contrary, let us assume that $\hat{z} \in \mathcal{Z}$ is not an LU-solution of (IVOP). Therefore, there exists some $z \in \mathcal{Z}$, such that

$$\Upsilon(z) \leq_{LU} \Upsilon(\hat{z}).$$

Without loss of generality, let

$$\underline{\Upsilon}(z) \leq \underline{\Upsilon}(\hat{z}), \qquad \overline{\Upsilon}(z) \leq \overline{\Upsilon}(\hat{z}). \tag{III.4}$$

Since $(\hat{z}, \hat{t}, \hat{u})$ is a solution of ACOP, it follows that:

$$\overline{\Upsilon}(\hat{z}) \leq \hat{t}, \qquad \underline{\Upsilon}(\hat{z}) \leq \hat{u}, \qquad \hat{u} \leq \hat{t}. \tag{III.5}$$

Combining (III.4) and (III.5) implies

$$\underline{\Upsilon}(z) < \underline{\Upsilon}(\hat{z}) \leq \overline{\Upsilon}(\hat{z}) \leq \hat{t}. \tag{III.6}$$

By defining $t := \hat{t}$, $u := \underline{\Upsilon}(z)$ and using (III.4), (III.5), (III.6), it follows:

$$\overline{\Upsilon}(z) \leq t, \qquad \underline{\Upsilon}(z) = u, \qquad u \leq t, \tag{III.7}$$

Hence, $(z, t, u) \in S_{\Upsilon}$. Considering (III.6) and (III.7), the following conclusion is obtained

$$t + u < \hat{t} + \hat{u}, \tag{III.8}$$

which leads to a contradiction.

Corollary 2 If $(\hat{z}, \hat{t}, \hat{u})$ is a local solution to the problem ACOP, then $\hat{z}$ is the local LU-solution of the problem IVOP.

Proof. The proof follows along similar lines as the proof of Theorem III.2.

Remark III.4 If $\hat{z} \in \mathcal{Z}$ is an LU-solution of IVOP, then there is no guarantee that there exist $\hat{t}, \hat{u} \in \mathbb{R}$, such that $(\hat{z}, \hat{t}, \hat{u})$ is a solution for problem ACOP. To illustrate this fact, we provide the following example.

Example III.4 Consider the following interval-valued optimization problem:

$$\text{(PIII.4.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := (-1, 1) \times (-1, 1)$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := [\underline{\Upsilon}(z_1, z_2), \overline{\Upsilon}(z_1, z_2)] = \left[\, z_1{+}z_2 \vee (z_1{+}z_2)^2 \,\right],$$

The corresponding associated scalar optimization problem ACOP is formulated in the following manner:

$$\text{(PIII.4.2)} \qquad \text{Minimize} \quad t + u,$$
$$\text{subject to} \quad \overline{\Upsilon}(z_1, z_2) \leq t,$$
$$\underline{\Upsilon}(z_1, z_2) \leq u,$$
$$u \leq t,$$
$$(z_1, z_2) \in \mathcal{Z}, \qquad t, u \in \mathbb{R}.$$

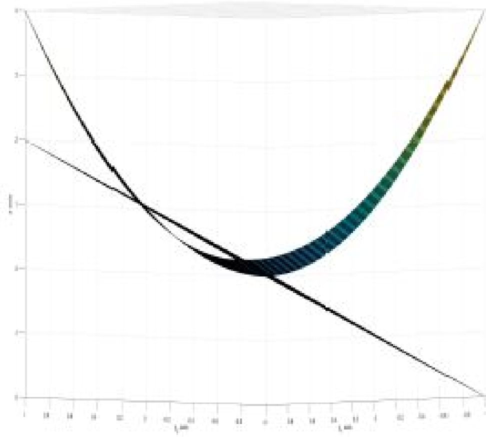

Fig. 4. $\Upsilon(x_1, x_2)$

From the Figure 4, it can be easily verified that $\hat{z} = (0, 0)$ is an LU-solution of the problem PIII.4.1. Suppose that there exist $\hat{t}, \hat{u} \in \mathbb{R}$, such that $((0, 0), \hat{t}, \hat{u})$ is a solution of Problem PIII.4.2. Therefore,

$$\hat{t} \geq \overline{\Upsilon}(0, 0) = 0 \qquad \text{and} \qquad \hat{u} \geq \underline{\Upsilon}(0, 0) = 0.$$

which leads to

$$\hat{t} + \hat{u} \geq 0.$$

On the other hand,

$$\left((0, -\tfrac{1}{2}), \tfrac{1}{4}, -\tfrac{1}{2}\right) \in S_{\Upsilon} \qquad \text{and} \qquad \tfrac{1}{4} + \left(-\tfrac{1}{2}\right) < \hat{t} + \hat{u},$$

which is a contradiction. Hence, there does not exist $\hat{t}, \hat{u} \in \mathbb{R}$, such that $((0, 0), \hat{t}, \hat{u})$ is a solution of the Problem PIII.4.2.

In the following example, the significance of Theorem III.2 is demonstrated. **Example III.5** Consider the following interval-valued optimization problem:

$$\text{(PIII.5.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := \mathbb{R}^2$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := [\underline{\Upsilon}(z_1, z_2), \overline{\Upsilon}(z_1, z_2)] = \left[z_1 \sin z_2,\ z_1^2{+}1\right], \qquad \text{for every (}$$

Therefore, $\underline{\Upsilon} : \mathcal{Z} \to \mathbb{R}$ and $\overline{\Upsilon} : \mathcal{Z} \to \mathbb{R}$ are defined as follows:

$$\underline{\Upsilon}(z_1, z_2) := z_1 \sin z_2,$$
$$\overline{\Upsilon}(z_1, z_2) := z_1^2 + 1, \qquad \text{for every } (z_1, z_2) \in \mathcal{Z}.$$

The ACOP corresponding to PIII.5.1 is given by PIII.5.2 as follows:

$$\text{(PIII.5.2)} \qquad \text{Minimize} \quad t + u,$$

$$\text{subject to} \quad z_1^2 + 1 \leq t,$$
$$z_1 \sin z_2 \leq u,$$
$$u \leq t.$$

After solving PIII.5.2 using the optimization tool called "fminproblem" in MATLAB R2024a, we get $\hat{z} = (0.5000, -1.5708)$, $\hat{t} = 1.2500$ and $\hat{u} = -0.5000$. Consequently, according to Theorem III.2, we conclude that $\hat{z} = (0.5000, -1.5708)$ serves as an LU-solution for PIII.5.1.

B. Optimality conditions for IVOP: gH-derivative approach

In this subsection, the necessary and sufficient conditions for the optimality of IVOP are established under appropriate assumptions. In the following theorem, necessary optimality conditions for the existence of an LU-solution of IVOP are presented.

Theorem III.3 Let $\hat{z} \in \mathcal{Z}$ and let $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ be an interval-valued function, such that $\nabla_{gH}\Upsilon(z)$ exists for all $z \in \mathcal{Z}$. If $\hat{z}$ is an LU-solution of (IVOP), then $0 \in \nabla_{gH}\Upsilon(\hat{z})$.

Proof. On the contrary, let us assume that $0 \notin \nabla_{gH}\Upsilon(\hat{z})$. Consequently, there exists an $i \in \mathcal{E}_n$, such that:

$$0 \notin \frac{\partial_{gH}\Upsilon(\hat{z})}{\partial z_i}.$$

Since $\mathcal{Z}$ is an open subset of $\mathbb{R}^n$ and $\hat{z} \in \mathcal{Z}$, therefore there exists $(z_1, \bar{z}_1) \times (z_2, \bar{z}_2) \times \cdots \times (z_n, \bar{z}_n)$, such that:

$$\hat{z} \in (z_1, \bar{z}_1) \times (z_2, \bar{z}_2) \times \cdots \times (z_n, \bar{z}_n) \subseteq \mathcal{Z},$$

where $\hat{z} = (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n)$.

Define $T : (z_i, \bar{z}_i) \to \mathcal{I}(\mathbb{R})$ as follows:

$$T(z) := \Upsilon(\hat{z}_1, \ldots, z, \ldots, \hat{z}_n), \qquad \text{for every } z \in (z_i, \bar{z}_i).$$

Since Υ has the i^{th} partial gH-derivative, therefore T is gH-differentiable at $\hat{z}_i$ and we have:

$$0 \notin T'_{gH}(\hat{z}_i) = \frac{\partial_{gH}\Upsilon(\hat{z})}{\partial z_i}.$$

Thus, there exists $\epsilon > 0$ and $\mathcal{E}(h) : (-\epsilon, \epsilon) \to \mathcal{I}(\mathbb{R})$, such that:

$$T(\hat{z}_i + h) \ominus_{gH} T(\hat{z}_i) = h \odot T'_{gH}(\hat{z}_i) \oplus \mathcal{E}(h),$$

where $\dfrac{\mathcal{E}(h)}{h} \to [0, 0]$ as $h \to 0$. Consequently, there exists a positive value of δ, such that one of the following holds:

1) For all $0 < h < \delta$, we have

$$T(\hat{z}_i + h) \ominus_{gH} T(\hat{z}_i) <_{LU} [0, 0],$$

2) For all $0 > h > -\delta$, we have

$$T(\hat{z}_i + h) \ominus_{gH} T(\hat{z}_i) <_{LU} [0, 0].$$

From Case 1, we get $h \in (0, \epsilon)$, such that:

$$\Upsilon(\hat{z}_1, \ldots, \hat{z}_i + h, \ldots, \hat{z}_n) <_{LU} \Upsilon(\hat{z}_1, \ldots, \hat{z}_i, \ldots, \hat{z}_n).$$

From Case 2, we get $h \in (-\epsilon, 0)$, such that:

$$\Upsilon(\hat{z}_1, \ldots, \hat{z}_i + h, \ldots, \hat{z}_n) <_{LU} \Upsilon(\hat{z}_1, \ldots, \hat{z}_i, \ldots, \hat{z}_n).$$

Hence, in both cases, we conclude that for any $\epsilon > 0$, there exists $h \in (-\epsilon, \epsilon)$, such that:

$$\Upsilon(\hat{z}_1, \ldots, \hat{z}_i + h, \ldots, \hat{z}_n) <_{LU} \Upsilon(\hat{z}_1, \ldots, \hat{z}_i, \ldots, \hat{z}_n),$$

which is a contradiction. This completes the proof.

Remark III.5 It is important to note that Theorem III.3 serve as an extension of Theorem 7 presented in Osuna-Gómez et al. [**?**], where the objective function of IVOP is defined on a subset of $\mathbb{R}$.

Remark III.6 The converse of Theorem III.3 is not always true. This fact is illustrated in the following example.

Example III.6 Let us consider the following interval-valued optimization problem:

$$\text{(PIII.6.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := (-1, 1) \times (-1, 1)$ and define $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ as follows:

$$\Upsilon(z_1, z_2) := \big[-(z_1 + z_2)^2, \ \cos(z_1 + z_2)\big].$$

It is observed that $\hat{z} = (0, 0)$ is not an LU-solution of Problem PIII.6.1, despite the fact that $0 \in \nabla_{gH}\Upsilon(\hat{z})$. This demonstrates that the converse of Theorem III.3 does not always hold.

In the following theorem, the necessary and sufficient optimality conditions for the existence of an LU-solution of IVOP are established. **Theorem III.4** Let $\mathcal{Z}$ be a convex subset of $\mathbb{R}^n$, and let $\hat{z} \in \mathcal{Z}$. Suppose that Υ is a strongly convex function defined on $\mathcal{Z}$ and possesses all gH-directional derivatives on $\mathcal{Z}$. Then, $\hat{z}$ is an LU-solution of IVOP if and only if

$$0 \in D_{gH}\Upsilon(\hat{z}; d)$$

for every nonzero direction $d \in \mathbb{R}^n$. *Proof.* Let $\hat{z}$ be an LU-solution of IVOP. Further, suppose that there exists a non-zero vector $d \in \mathbb{R}^n$, such that $0 \notin D_{gH}\Upsilon(\hat{z}; d)$. Consequently, one of the following holds:

$$\text{Case (1)} \qquad D_{gH}\Upsilon(\hat{z}; d) <_{LU} [0, 0],$$
$$\text{Case (2)} \qquad [0, 0] <_{LU} D_{gH}\Upsilon(\hat{z}; d).$$

For Case (1), there exists $\alpha_1 > 0$, such that:

$$\Upsilon(\hat{z} + \alpha d) \leq_{LU} \Upsilon(\hat{z}), \qquad \text{for every } \alpha \in (0, \alpha_1).$$

For Case (2), there exists $\alpha_2 > 0$, such that:

$$\Upsilon(\hat{z} + \alpha d) \leq_{LU} \Upsilon(\hat{z}), \qquad \text{for every } \alpha \in (-\alpha_2, 0).$$

Thus, in both cases, it can be concluded that for any $\epsilon > 0$, there exists $\alpha \in (-\epsilon, \epsilon)$, such that:

$$\Upsilon(\hat{z} + \alpha d) \leq_{LU} \Upsilon(\hat{z}),$$

which contradicts the fact that $\hat{z}$ is an LU-solution of IVOP.

Conversely, let $\hat{z}$ is not an LU-solution of IVOP. This implies that there exists $z \in \mathcal{Z}$, such that:

$$\Upsilon(z) \leq_{LU} \Upsilon(\hat{z}). \tag{III.9}$$

Define $d := z - \hat{z}$. Since Υ is strongly convex, there exists $\gamma > 0$, such that for any $\mu \in (0, 1)$, the following inequality holds:

$$\Upsilon((1-\mu)\hat{z} + \mu z) \oplus \mu \|z - \hat{z}\|^2 \odot [\gamma, \gamma] \leq_{LU} (1-\mu) \odot \Upsilon(\hat{z}) \oplus \mu \odot \Upsilon(z).$$

As a result, the following conclusion is obtained:

$$\underline{\Upsilon}((1-\mu)\hat{z}+\mu z)+\mu\|z-\hat{z}\|^2\gamma \leq (1-\mu)\underline{\Upsilon}(\hat{z})+\mu\underline{\Upsilon}(z),$$

$$\overline{\Upsilon}((1-\mu)\hat{z}+\mu z)+\mu\|z-\hat{z}\|^2\gamma \leq (1-\mu)\overline{\Upsilon}(\hat{z})+\mu\overline{\Upsilon}(z).$$

Hence, it follows that:

$$\frac{\Upsilon(\hat{z} + \mu d) \ominus_{gH} \Upsilon(\hat{z})}{\mu} \oplus \|z - \hat{z}\|^2 \odot [\gamma, \gamma] \leq_{LU} \tag{IV.1}$$

$$\Upsilon(z) \ominus_{gH} \Upsilon(\hat{z}). \tag{IV.2}$$

In view of (III.9) and by letting $\mu \to 0$, the following is obtained:

$$D_{gH}\Upsilon(\hat{z}; d) <_{LU} [0, 0],$$

which is a contradiction.

Corollary 3 Suppose that $\mathcal{Z}$ is convex and $\hat{z} \in \mathcal{Z}$. If Υ is local strongly convex at $\hat{z}$ and has all the gH-directional derivatives on $\mathcal{Z}$, then $\hat{z}$ is a local LU-solution of IVOP if and only if the following condition holds for every non-zero $d \in \mathbb{R}^n$:

$$0 \in D_{gH}\Upsilon(\hat{z}; d).$$

Proof. The proof follows along similar lines as the proof of Theorem III.4.

Remark III.7 If the function Υ is assumed to be convex (rather than strongly convex), then the consequences of Theorem III.4 fail to be satisfied. To illustrate this fact, we provide the following example. **Example III.7** Consider the following interval-valued optimization problem:

$$\text{(PIII.7.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2),$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := \mathbb{R}^2$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := [\underline{\Upsilon}(z_1, z_2),\ \overline{\Upsilon}(z_1, z_2)] := [0,\ z_1^2+z_2^2], (z_1, z_2) \in \mathcal{Z}.$$

It is easy to see that, the function Υ is convex and we have $0 \in \nabla_{gH}\Upsilon\left(0, \frac{1}{2}\right)$. However, despite this fact, $(0, \frac{1}{2})$ is not an LU-solution of PIII.7.1.

Now, a numerical example is provided to illustrate the significance of Theorem III.4. **Example III.8** Consider the following interval-valued optimization problem:

$$\text{(PIII.8.1)} \qquad \text{Minimize} \quad \Upsilon(z_1, z_2) = [\underline{\Upsilon}(z_1, z_2),\ \overline{\Upsilon}(z_1, z_2)],$$
$$\text{subject to} \quad (z_1, z_2) \in \mathcal{Z},$$

where $\mathcal{Z} := (-1, 1) \times (-1, 1)$ and $\Upsilon : \mathcal{Z} \to \mathcal{I}(\mathbb{R})$ is defined as follows:

$$\Upsilon(z_1, z_2) := \left[z_1^2+z_2^2,\ z_1^4+e^{4z_2}+3\right], \qquad \text{for every } (z_1, z_2) \in \mathcal{Z}.$$

It can be verified that $0 \in D_{gH}\Upsilon((0, 0), d)$, for every $d \in \mathbb{R}^n$ and Υ is strongly convex on $\mathcal{Z}$. Therefore, according to Theorem III.4, $(0, 0)$ is an LU-solution of PIII.8.1.

V. CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

In this paper, a specific class of IVOP has been investigated. The associated ASOP and ACOP were formulated, and it was established that any optimal solution of either ASOP or ACOP corresponds to an LU-solution of IVOP. Additionally, necessary and sufficient optimality conditions for IVOP were derived under appropriate assumptions on the objective function. To demonstrate the practical relevance of the theoretical results, several non-trivial numerical examples were presented. The findings of this paper extend existing optimality results in the literature. In particular, the optimality conditions for IVOP established by Osuna-Gómez et al. [**?**] have been generalized from the domain of real numbers to the broader framework of Euclidean spaces.

The results established in this paper open numerous avenues for future research. For instance, deriving second-order necessary and sufficient optimality conditions for IVMOP presents an exciting research direction. This will be pursued as part of future work.

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