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A penalty approach to solve MPCC problems

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***Abstract*—We consider a new approach that combines penalization and relaxation techniques to solve the MPCC (Mathematical Program with Complementarity Constraints). We prove under the MPCC-Mangasarian-Fromovitz constraints qualifications that any accumulation point of the approximate solution sequence produced by our method is an M-stationary point of the original MPCC. We present some numerical experiments on problems from the MacMPEC library to confirm the efficiency of our approach.**

***Index Terms*—Mathematical programming with complementarity constraints, Penalty function, Regularization techniques.**

I. INTRODUCTION

We consider the Mathematical Program with Complementarity Constraints:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & f(x) \\ \text{s.t.} & h(x)=0,\ g(x)\le 0, \\ & 0\le G(x)\perp H(x)\ge 0, \end{array} \qquad \text{(MPCC)}$$

with $f:\mathbb{R}^n\to\mathbb{R}, g:\mathbb{R}^n\to\mathbb{R}^q, h:\mathbb{R}^n\to\mathbb{R}^p$, and $G,H:\mathbb{R}^n\to\mathbb{R}^m$. All these functions are assumed to be continuously differentiable through this paper. The notation $0\le u\perp v\ge 0$ for two vectors u and v in $\mathbb{R}^q$ is a shortcut for $u\ge 0, v\ge 0$ and $u^Tv=0$.

MPCC (Mathematical Programming with Complementarity Constraints) is an important class of problems. It arises frequently in applications in engineering design, economic equilibrium, and multilevel games [23]. MPCC also comes from bilevel programming problems, which have numerous applications in practice [32].

The MPCC is a challenging subclass of non-linear programming problems due to the degeneracy of complementarity constraints. Indeed, the classical Mangasarian-Fromovitz that is very often used to guarantee convergence of algorithms is violated at any MPCC feasible point. This is partly due to the geometry of the complementarity constraint that always has an empty relative interior. These issues have motivated the definition of enhanced constraints qualifications and optimality conditions for (MPCC) as in [10], [11]. In 2005, Flegel and Kanzow [11] defines the "right" necessary optimality condition to (MPCC), it called M(Mordukhovich)-stationary condition. In the literature, there exist a wide variety of MPCC constraints and qualifications, we will focus, in this study, on the MPCC-Mangasarian Fromovitz CQs (MPCC-MFCQ).

A wide range of numerical methods have been proposed to solve this problem such as relaxation methods [8], [9], interior-point methods [21], [25], [30], penalty methods [15], [23], [28], SQP methods [12], dc methods [26], filter methods [20] and Levenberg-Marquardt methods [13]. Among these methods, the relaxation is one of the most popular approaches, which consists of relaxing the complementarity constraints using a parameter, then generates a sequence of non-linear programs, which are more regular than the initial problem. Thus, one can apply the well-studied numerical methods for non-linear programming. Therefore, it is often necessary to prove theoretically that the sequence of approximate solutions converges to a stationary point (or an solution) of the original MPCC.

In [4], [5] the authors have proposed a regularization method which ensures global convergence to stationary point for MPCC with bound constraints. In [18], Kanzow and Schwartz discussed the convergence of the relaxation methods by considering a sequence of approximate stationary points. They proved that these methods may converge to spurious weak-stationary points. In [7], Azizi and Kadrani proposed an approach based on penalty formulation and a relaxation scheme for MPCC, they showed that any accumulation point of the sequence of strong approximate stationary points is a M-stationary point for the MPCC.

We present in this paper a new regularization-penalization method to solve the MPCC : firstly we regularize the complementarity constraints by using concave and nondecreasing functions θ introduced in [14]:

$0\le x\perp y\ge 0,\quad$ is relaxed to $\quad\theta(t,x_i)+\theta(t,y_i)\le 1,$
$i=1,\dots,n.$

Then we modify the regularization-penalization scheme [16] by using another penalty function. Our objective is to propose a new algorithm that converges to an M-stationary point of the MPCC under the MPCC-Mangasarian-Fromovitz constraints qualifications.

In [16], the authors introduced the parametric barrier $\frac{\Delta(z,t)}{1-k\Delta(z,t)}$ to define the penalty function. In our approach, we will consider similar penalty term but we will not need any complicated strategy to update the regularization parameter t since we will consider it as a new variable.

The remainder of the paper is organized as follows. In the next section, we introduce some basic concepts from mathematical programming with complementarity constraints. In Section 3, we present our approximation and formulation. Section 4, is devoted to the convergence proof. In Section 5, numerical experiments are reported on some examples from the MacMpec tests library [19].

Throughout this paper, G_i represents the i-th component of a vector G and similar notations are used for vector-valued functions. ∇f denotes the gradient of a differentiable real value function f defined on $\mathbb{R}^n$. The norm $\|.\|$ denotes the Euclidian norm.

II. Definitions for MPCC and Preliminaries

A. Definitions for MPCC

In this section, we give some definitions and fix some notations for the rest of the paper. Let $\mathcal{Z}$ be the set of feasible points of (MPCC). Given, $x \in \mathcal{Z}$, we denote

$$\begin{aligned}
&\mathcal{I}^{+0}(x) = \{i \in \{1,\dots,q\} \mid G_i(x) > 0 \text{ and } H_i(x) = 0\},\\
&\mathcal{I}^{0+}(x) = \{i \in \{1,\dots,q\} \mid G_i(x) = 0 \text{ and } H_i(x) > 0\},\\
&\mathcal{I}^{00}(x) = \{i \in \{1,\dots,q\} \mid H_i(x) = 0 \text{ and } G_i(x) = 0\},\\
&\mathcal{I}_g(x) = \{i \in \{1,\dots,p\} \mid g_i(x) = 0\}.
\end{aligned}$$

We define the generalized MPCC-Lagrangian function of (MPCC) as

$$\mathcal{L}^r_{MPCC}(x,\lambda) = rf(x) + \lambda^g g(x) + \lambda^h h(x) - \lambda^G G(x)$$
$$-\lambda^H H(x),$$

and denote $\lambda := (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$.

In general, MPCC does not have any KKT stationary point. So we need weaker stationary concepts as in [8], [9].

Definition 2.1 (Stationary point): $x^* \in \mathcal{Z}$ is said

- Weak-stationary if there exist $\lambda \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^q$ such that
$$\begin{aligned}
&\nabla_x \mathcal{L}^1_{MPCC}(x^*,\lambda) = 0,\\
&\min(g(x^*),\lambda^g) = 0, h(x^*) = 0,\\
&\forall i \in \mathcal{I}^{+0},\ \lambda_i^G = 0, \text{ and } \forall i \in \mathcal{I}^{0+},\ \lambda_i^H = 0;
\end{aligned}$$
- Clarke-stationary point if x^* is weak-stationary and
$$\forall i \in \mathcal{I}^{00},\ \lambda_i^G \lambda_i^H \geq 0;$$
- Alternatively (or Abadie)-stationary point if x^* is weak-stationary and
$$\forall i \in \mathcal{I}^{00},\ \lambda_i^G \geq 0 \quad \text{or} \quad \lambda_i^H \geq 0;$$
- Mordukhovich-stationary point if x^* is weak-stationary and
$$\forall i \in \mathcal{I}^{00},\ \text{either} \quad \lambda_i^G > 0, \lambda_i^H > 0 \quad \text{or} \quad \lambda_i^G \lambda_i^H = 0;$$
- Stong-stationary point if x^* is weak-stationary and
$$\forall i \in \mathcal{I}^{00},\ \lambda_i^G \geq 0, \quad \lambda_i^H \geq 0.$$

It is also to be noted that if we assume strict complementarity, i.e. for all $i \in \{1,\dots,q\}$

$$H_i(x) + G_i(x) > 0$$

then all of the stationary conditions presented here are equivalent. For instance, this is the case for the class of binary optimisation problems, whose integer constraints are replaced by complementarity constraints.

There exist a wide variety of MPCC constraint qualification described in the literature. We conclude this section by defining the only one needed later, the MPCC-Mangasarian Fromovitz CQs (MPCC-MFCQ).

Definition 2.2: Let $x^* \in \mathcal{Z}$. MPCC-MFCQ holds at x^* if

$$\sum_{i\in\mathcal{I}_g(x^*)} \alpha_i \nabla g_i(x^*) + \sum_{i=1}^{p} \beta_i \nabla h_i(x^*) + \sum_{i\in\mathcal{I}^{00}(x^*)\cup\mathcal{I}^{0+}(x^*)} \gamma_i \nabla G_i(x^*)$$

$$+ \sum_{i\in\mathcal{I}^{00}(x^*)\cup\mathcal{I}^{+0}(x^*)} \delta_i \nabla H_i(x^*) = 0$$

with $\alpha_i \geq 0, \delta_i \geq 0$ and $\gamma_i \geq 0$. We have $\alpha = \gamma = \delta = 0$.

In view of the constraint qualifications issues that plague the (MPCC), we regularize the complementarity constraints by using some smoothing functions. We present now how we construct this functions.

III. Presentation of γ and θ smoothing functions

We consider a family of smooth functions that are extensively used in various numerical methods. These functions are non-decreasing continuous smooth (C^1) concave functions such that

$\gamma : \mathbb{R} \to]-\infty, 1[$ with $\gamma(x) < 0$ if $x < 0, \gamma(0) = 0,$ and $\lim_{x\to+\infty} \gamma(x) = 1.$

One generic way to build such functions is to consider non-increasing continuous probability density functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ and then take the corresponding cumulative distribution functions

$$\forall x \geq 0, \quad \gamma(x) = \int_0^x f(s)ds.$$

By definition of f

$$\lim_{x\to+\infty} \gamma(x) = \int_0^{+\infty} f(s)ds = 1 \text{ and}$$
$$\gamma(0) = \int_0^0 f(s)ds = 0.$$

The hypothesis on f gives the concavity of θ.

We introduce $\theta(t, x) := \gamma(\frac{x}{t})$ for $t > 0$. This definition is similar to the perspective functions in convex analysis. These functions satisfy $\theta(t, 0) = 0$, and $\forall x > 0, \lim_{t\searrow 0} \theta(t, x) = 1$.
In order to simplify the presentation, we sometimes used the notation $\theta(t, x)$ instead of $\theta_t(x)$. We denote ∂_x or ∂_t the derivative with respect x and t respectively.

From now on, we assume that:

$$\lim_{t\searrow 0} \theta(t, \sqrt{t}) = 1 \quad \text{and} \quad \lim_{t\searrow 0} \partial_x\theta(t, \sqrt{t}) > 0. \qquad (A)$$

Remark 3.1: All these assumptions are not restrictive. Indeed the two functions θ^1 and θ^2 defined below (and all functions in between them) satisfy (A).

$$\forall x \geq 0, \qquad \theta^1(t, x) := \frac{x}{x + t}, \quad \text{and} \quad \theta^2(t, x) := 1-e^{-\frac{x}{t}},$$

and for $x < 0$, $\theta^{1,2}$ is a linear function $\theta^{1,2}(t, x) = ax$ where $a = \partial_x\theta^{1,2}(t, 0) > 0$.
The corresponding derivatives with respect to x are
$\partial_x\theta^1(t, \sqrt{t}) = \frac{1}{(1+\sqrt{t})^2}$ and $\partial_x\theta^2(t, \sqrt{t}) = \frac{1}{t}e^{-\frac{1}{\sqrt{t}}}$.

A. A θ-smoothing of a complementarity condition

In this paper, we use the θ-functions to regularize the complementarity constraints. The following obvious lemma provides an intuition of the motivation behind such technique.

Lemma 3.2: [4] Given $x, y \in \mathbb{R}_+$, we have

$$x \perp y \Longleftrightarrow \lim_{t\searrow 0} \theta(t, x) + \theta(t, y) \leq 1$$

We prove now some technical lemmas and results that are used in the sequel of the paper to proof the convergence theorem.

Lemma 3.3: $\forall x > 0$, we have

$$\lim_{t\searrow 0} \ \partial_x\theta(t, x) = 0.$$

We have, $\partial_x\theta(t, x) = \frac{1}{t}\gamma^{'}(\frac{x}{t})$. So,

$$\lim_{t\searrow 0} \ x.\partial_x\theta(t, x) = \lim_{z\longrightarrow\infty} z\gamma^{'}(z).$$

Let $s > s^{'}$, then by the mean value theorem there exist $c \in [s, s^{'}]$ such that $\gamma(s) - \gamma(s^{'}) = \gamma^{'}(c)(s - s^{'}) \geq 0$.
By the concavity of γ it follows $\gamma(s) - \gamma(s^{'}) \geq \gamma^{'}(s)(s - s^{'})$.
By taking, $s^{'} = \frac{s}{2}$, we have

$$2(\gamma(s) - \gamma(\frac{s}{2})) \geq \gamma^{'}(s)s \geq 0.$$

Using that $\lim_{z\longrightarrow\infty} \gamma(z) = 1$ and passing to the limit, we obtain that

$$\lim_{t\searrow 0} \ x.\partial_x\theta(t, x) = 0.$$

The result follows since $x > 0$.

Lemma 3.4: Let $t_0 > 0$. For every $t \in (0, t_0]$, we have the two following results:

(i) The set $\{\partial_x\theta(t, x), x \in \mathbb{R}\}$ is bounded, and

(ii) there exist two positive numbers M and M' such that the set $\{\partial_t\theta(t, x), x \geq -t_0\}$ is bounded by $\max\left(\frac{M}{t}, \frac{M^{'}}{t^2}\right)$.

(i) θ is concave with respect x, so the derivative with respect x is decreasing.
On the other hand, θ is increasing so the derivative with respect x is positive. So for $x \geq 0$,

$$0 \leq \partial_x\theta(t, x) \leq \partial_x\theta(t, 0).$$

For $x \leq 0$, we have $\theta_t(x) = \alpha_t x$, with $\alpha_t = \partial_x\theta(t, 0)$. So we have the result.

(ii) We have shown that $\lim_{z\longrightarrow\infty} z\gamma^{'}(z) = 0$. Then, the set $\{z\gamma^{'}(z), z \in [0, +\infty[\}$ is bounded, so there exist $M > 0$ such that $|z\gamma^{'}(z)| < M$, for $z \in [0, +\infty[$.
The derivative with respect t is : $|\partial_t\theta(t, x) = \partial_t\gamma(\frac{x}{t})| = | - \frac{x}{t^2}\gamma^{'}(\frac{x}{t})|$.

We consider two cases: When $x \geq 0, |\partial_t\theta(t,x) = |-\frac{1}{t}\frac{x}{t}\gamma^{'}(\frac{x}{t})| \leq \frac{M}{t}$.

When $-t_0 \leq x < 0$, $|\partial_t\theta(t,x)| = \gamma^{'}(0)$.
So, for $M' = t_0\gamma^{'}(0)$, we have $|\partial_t\theta(t,x)| = |-\frac{x}{t^2}\gamma^{'}(0)| \leq \frac{M'}{t^2}$.

These functions are further extended as functions from $\mathbb{R}^n$ in $\mathbb{R}^n$ component by component, i.e. for a vector $x \in \mathbb{R}^n$ we have $\theta(t,x) = (\theta(t,x_i))_{1\leq i\leq n}$.

IV. A Penalization-Regularization Approach

The first attempt in the literature to use a relaxation technique for MPCC problems goes back to S. Scholtes in 2001 [31]. In this paper, the author used the following approximation of the complementarity condition

$$\Phi_i^{SS}(G(x),H(x);t) := G_i(x)H_i(x) - t^2; \qquad (SS)$$

It is now, will known that this method converges to a C-stationary point if MPCC-MFCQ holds at its limit point. This relaxation is clearly more regular than (MPCC) since MFCQ is violated at any feasible point of the original problem. The idea to additionally relax the positivity constraints (i.e. consider $\bar{t} > 0$ such that $G(x) \geq -\bar{t}e, H(x) \geq -\bar{t}e$) has been introduced in [27] as an extension to the relaxation (SS).
The relaxation (SS) is very unlikely to satisfy LICQ at a point x^* with $\mathcal{I}^{00}(x^*) \neq \emptyset$, since this would mean that for $t = 0$ three constraints are active for only two gradients.
$G.-H.$ Lin and M. Fukushima proposed in [22] a relaxation with fewer constraints in order to improve the regularity of the relaxed program by considering:

$$\begin{array}{ll}\min\limits_{x\in\mathbb{R}^n} & f(x)\\ \text{s.t.} & h(x)=0,\ g(x)\geq 0,\\ & G_i(x)H_i(x)\leq t^2, i=1,\dots,m\\ & (G_i(x)+t)(H_i(x)+t)\geq t^2, i=1,\dots,m\end{array} \qquad (P^{LF})$$

The authors showed in particular that if MPCC-LICQ holds at a point $x^* \in \mathcal{Z}$, this relaxation satisfies the classical LICQ in any feasible points of P^{LF} in a neighborhood of x^*. Once again, this relaxation converges to a C-stationary point as $t \searrow 0$ as shown in [17].
In [5], Abdallah et al. used the family of smoothing functions introduced in section 3 to approximate the complementarity condition in the simple where $H(x) = x, G(x) = y$ as follows

$$\Phi_i(x,y;t) := \theta(t,x_i) + \theta(t,y_i) - 1, \quad i = 1,\dots,n.$$

with $\Phi(x,y;t) = (\Phi_i(x,y;t))_{1\leq i\leq n}$. In this paper they only considered bound constraints and not general MPCC.

A. Our formulation

In this section, we present our formulation to solve the MPCC problem. We first introduce slack variables u and v to the general nonlinear constraints of the MPCC problem:

$$u = G(x), \quad v = H(x), u \geq 0, v \geq 0.$$

and the slack variables s for the constraints $g(x)$. We obtain

$$\begin{array}{l}\min\limits_{z} f(x)\\ \text{s.t. } g(x)-s=0,\ h(x)=0, s\geq 0,\\ \quad H(x)-u=0, \quad G(x)-v=0,\\ \quad 0\leq u\perp v\geq 0.\end{array} \qquad \text{(IV.1)}$$

with $z = (x,u,v,s)$.
This problem can be written as follows

$$\begin{array}{l}\min\limits_{z} f(x)\\ \text{s.t. } F(z)=0,\\ \quad s\geq 0,\ 0\leq u\perp v\geq 0.\end{array} \qquad (\widetilde{1})$$

where $F(z) := \begin{pmatrix} g(x)-s\\ h(x)\\ H(x)-u\\ G(x)-v\end{pmatrix}$.

This problem has the same properties as the original MPCC problem, in the sense that if x satisfies some MPCC-CQs, and some weak weak or strong stationary conditions for MPCC, the point z satisfies these conditions for $(\widetilde{1})$.

We regularize the complementarity constraints by using continuously, differentiable, non-decreasing and concave θ functions and by the lemma 3.2. We also relax the inequality constraints $v \geq 0$. Consequently, the original MPCC is approximated by the following new smooth parametrized optimization problem:

$$\begin{array}{l}\min\limits_{z} f(x)\\ \text{s.t. } F(z)=tw,\\ \quad \Phi_l(u,v;t)\leq 0, l=1,\dots,m,\\ \quad u\geq 0, v\geq -t, s\geq 0.\end{array} \qquad (2_t)$$

where $w = (1,\dots,1) \in \mathbb{R}^{p+q+2m}$.

At the limit when the relaxation parameter t is driven to zero, the feasible set of the parametric non-linear program (2_t) converges to the feasible set of the $(\widetilde{1})$. Let $\mathcal{Z}_t$ be the feasible set of (2_t), and $\mathcal{Z}$ the feasible set of $(\widetilde{1})$, it holds that

$$\lim_{t\to 0} \mathcal{Z}_t = \mathcal{Z}.$$

(Every converging sequence of points in $\mathcal{Z}_t$ has its limit in $\mathcal{Z}$)
Now, we consider the penalty function [16] defined as follows

$$f_\sigma(z,t) := \begin{cases} f(x) & \text{if} \quad t = \Delta(z,t) = 0 \\ f(x) + \dfrac{\Delta(z,t)}{2t} + \sigma\beta(t) & \text{if} \quad t > 0, \\ +\infty & \text{otherwise} \end{cases} \tag{IV.2}$$

where $\Delta(z,t) := \|F(z) - tw\|^2$ measures the constraints violation. The function $\beta : [0,\bar{t}] \to [0,\infty)$ is continuously differentiable on $(0,\bar{t}]$ and $\beta(0) = 0$ (for some fixed $\bar{t}$). The term $\sigma\beta(t)$ allows to consider t as a new optimization variable, and minimize simultaneously z and t. In this study, we will take $\beta(t) = \sqrt{t}$.

Our penalized-regularized problem then

$$\begin{aligned} &\min f_\sigma(z,t) \\ &\text{s.t. } \Phi_l(u,v;t) \le 0, l = 1,\dots,m, \\ &\quad u \ge 0, v \ge -t, s \ge 0. \end{aligned} \tag{P_σ}$$

The Lagrangian function of (P_σ) is defined as follows:

$$\begin{aligned} \mathcal{L}_\sigma(z,t,\mu_1,\mu_2,\eta,\nu) := f_\sigma(z,t) - \nu^T s - (\mu_1)^T u \\ -(\mu_2)^T(v + t.\mathbf{1}) + \eta^T\Phi(u,v;t). \end{aligned}$$

Since it is usually not possible to solve P_σ exactly, we will consider a strong ϵ-stationary point of P_σ.

Definition 4.1: We say that $(z,t) \in \mathbb{R}^{n+2m+q} \times \mathbb{R}$ is a strong ϵ-stationary point of P_σ if there exist multipliers ν, μ_1, μ_2, η such that:

$$\|\nabla_z \mathcal{L}_\sigma(z,t)\|_\infty \le \epsilon$$

$$\begin{aligned} &| -\frac{1}{2}t^{-2}\Delta - t^{-1}\sum_{j=1}^{p+q+2m}(F_j - t) \\ &+ \sum_{i=1}^{m}(\mu_{2i} + \eta_i(\partial_t\theta(t,u_i) + \partial_t\theta(t,v_i))) + \sigma\frac{1}{2\sqrt{t}}| \le \epsilon \\ &s_i \ge -\epsilon \quad \nu_i \ge 0 \quad |s_i\nu_i| \le \epsilon \\ &u_l \ge -\epsilon \quad \mu_{1,l} \ge 0 \quad |\mu_{1,l}u_l| \le \epsilon \\ &v_l + t \ge -\epsilon \quad \mu_{2,l} \ge 0 \quad |\mu_{2,l}(v_l+t)| \le \epsilon \\ &0 \le \eta \perp \Phi(u,v;t) \le 0 \end{aligned} \tag{IV.3}$$

V. CONVERGENCE ANALYSIS

Before starting our analysis, we need to define different index sets for the active constraints. We will denote for fixed t.

$$\begin{aligned} I_0(z) &= \{i : s_i = 0\}, \\ I_1(z) &= \{l : u_l = 0\}, \\ I_2(z) &= \{l : v_l = -t\}, \\ I_\Phi(z) &= \{l : \Phi_l = 0\}. \end{aligned}$$

and

$$\begin{aligned} I_0^*(z^*) &= \{i : s_i^* = 0\}, \\ I_{0+}^*(z^*) &= \{l : u_l^* = 0, v_l^* > 0\}, \\ I_{+0}^*(z^*) &= \{l : u_l^* > 0, v_l^* = 0\}, \\ I_{00}^*(z^*) &= \{l : u_l^* = 0, v_l^* = 0\}. \end{aligned}$$

We start by two essential lemmas.

Lemma 5.1: For every $t > 0$, we have $I_1(z) \cap I_\Phi(z) = \emptyset$ and $I_2(z) \cap I_\Phi(z) = \emptyset$.
Suppose that $I_1(z) \cap I_\Phi(z) \ne \emptyset$, so there exist an index $i \in I_1(z) \cap I_\Phi(z)$ then $u_i = 0$ and $\Phi_i(u,v) = 0$ so $u_i = 0$ and $\theta(t,u_i) + \theta(t,v_i) - 1 = 0$. This implies that $\theta(t,v_i) = 1$ which impossible for $t > 0$.
We suppose now that that $I_2(z) \cap I_\Phi(z) \ne \emptyset$, so there exist $i \in I_2(z) \cup I_\Phi(z)$ then $v_i = -t$ and $\Phi_i(u,v) = 0$ (i.e. $v_i = -t$ and $\theta(t,u_i) + \theta(t,v_i) - 1 = 0$. This implies that $\theta(t,v_i) > 1$ this impossible for $t > 0$.

Lemma 5.2: Let (z^k, t_k) be a strong ϵ_k-stationary point sequence of the penalized-regularized problem with $\epsilon_k = o(t_k)$. If (z^*, t_*) is a cluster point of the sequence $\{(z^k, t_k)\}$, we have

$$\begin{aligned} &\forall r \in I_\Phi(z^k) \cap (I_{00}^*(z^*) \cup I_{+0}^*(z^*) \cup I_{0+}^*(z^*)), \\ &\lim_{t_k \longrightarrow t_*} \max\left(\partial_{u_r}\theta(t_k, u_r^k), \partial_{v_r}\theta(t_k, v_r^k)\right) > 0. \end{aligned}$$

If $t_* \ne 0$, there is nothing to prove, we only consider the case for $t_* = 0$.
Let $r \in I_\Phi(z^k) \cap I_{0+}^*(z^*)$, in this case, we have $u_r^k \le \sqrt{t_k}$. Indeed, if $u_r^k > \sqrt{t_k}$, so $\theta(t_k, u_r^k) > \theta(t_k, \sqrt{t_k})$ and $\theta(t_k, u_r^k) + \theta(t_k, v_r^k) > \theta(t_k, v_r^k) + \theta(t_k, \sqrt{t_k})$. we obtain $\theta(t_*, u_r^*) + \theta(t_*, v_r^*) > \theta(t_*, v_r^*) + \theta(t_*, \sqrt{t_*})$. This is a Contradiction. On the other hand, the function $\partial_{u_r}\theta(t,.)$ is decreasing since $\theta(t,.)$ is concave, so $\partial_{u_r}\theta(t_k, u_r^k) \ge \partial_{u_r}\theta(t_k, \sqrt{t_k}) > 0$.

For $r \in I_\Phi(z^k) \cap I_{+0}^*(z^*), \partial_{v_r}\theta(t_k, v_r^k) > 0$ (same proof as above).
For $r \in I_\Phi(z^k) \cap I_{00}^*(z^*)$. So, $u_r^k \le \sqrt{t_k}$ or $v_r^k \le \sqrt{t_k}$. Indeed, if $u_r^k > \sqrt{t_k}$ and $v_r^k > \sqrt{t_k}$, so $\theta(t_k, u_r^k) + \theta(t_k, v_r^k) > 2\theta(t_k, \sqrt{t_k})$. For t_k sufficiently small, in the neighborhood of t_* we obtain, $\theta(t_k, u_r^k) + \theta(t_k, v_r^k) > 2$. This is a Contradiction because $r \in I_\Phi(z^k)$. So, as the proof of the previous case, we obtain the result. We study now the relation between the standard qualification constraints of the problem (2_t) and the MPCC. Note that, the reformulation with the slacks variables for the

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$(\widetilde{1})$ does not alter the properties of the original MPCC problem. In particular, if we assume that x^* satisfies MPCC-MFCQ for (1), z^* satisfies also MPCC-MFCQ for $(\widetilde{1})$.

Theorem 5.3: Let x^* be a feasible point of the original problem (1) such that MPCC-MFCQ is satisfied at x^*. Then, there exist a neighborhood $\mathcal{U}(z^*)$ of z^* and a sufficiently small scalar $\bar{t} > 0$ such that for any $t \in (0, \bar{t})$, any $z \in \mathcal{U}(z^*) \cap \mathcal{Z}_t$ satisfies the standard MFCQ for the problem (2_t).
satisfies the standard MFCQ for any $z \in \mathcal{U}(z^*) \cap \mathcal{Z}_t$.

Since all the constraints functions of $(\widetilde{1})$ are continuously differentiable (the constraints functions of the MPCC problem are continuously differentiable) then, there exist a neighborhood $\mathcal{U}_1(z^*)$ and a positive scalar $\bar{t}_1$ such that $\forall t \in (0, \bar{t}_1)$ and for every $z \in \mathcal{U}_1(z^*) \bigcap \mathcal{Z}_t$ we have:

$$\begin{array}{l} I_0(z) \subseteq I_0^*(z^*) \\ I_1(z) \subseteq I_{00}^*(z^*) \cup I_{0+}^*(z^*) \\ I_2(z) \subseteq I_{00}^*(z^*) \cup I_{+0}^*(z^*) \end{array}$$

Since $(\widetilde{1})$-MFCQ is satisfied at z^* for the problem $(\widetilde{1})$, the vectors $\{\nabla F_j | j = 1, \dots p+q+2m\} \cup \{-e_{n+i} | i \in I_0^*(z^*)\} \cup \{e_{n+q+k} | k \in I_{00}^*(z^*) \cup I_{0+}^*(z^*)\} \cup \{e_{n+q+m+l} | l \in I_{00}^*(z^*) \cup I_{+0}^*(z^*)\}$ are positively linearly independent where e_l is the unit vector of size $n+q+2m$.

We have:

$$\begin{array}{l} I_1(z) \cup (I_\Phi(z) \cap I_{0+}^*(z^*)) \cup (I_\Phi(z) \cap I_{00}^*(z^*)) \\ \qquad \subseteq I_{00}^*(z^*) \cup I_{0+}^*(z^*) \\ I_2(z) \cup (I_\Phi(z) \cap I_{+0}^*(z^*)) \cup (I_\Phi(z) \cap I_{00}^*(z^*)) \\ \qquad \subseteq I_{00}^*(z^*) \cup I_{+0}^*(z^*) \end{array} \tag{V.1}$$

By the proposition 2.2 [29] there exist a neighborhood $\mathcal{U}_2(z^*)$ and a positive constant $\bar{t}_2$ sufficiently small such that for all points $z \in \mathcal{U}_2(z^*) \bigcap \mathcal{Z}_t$ with $t \in (0, \bar{t}_2)$, the following vectors

$$\begin{array}{l} \{\nabla F_j | j = 1, \dots p+q+2m\} \cup \ \{-e_{n+i} | i \in I_0(z)\} \\ \cup \{e_{n+q+k} | k \in I_1(z)\} \cup \ \{e_{n+q+m+l} | l \in I_2(z)\} \\ \cup \{e_{n+q+r} | r \in I_\Phi(z) \cap I_{0+}^*(z^*)\} \\ \cup \{e_{n+q+r} | r \in I_\Phi(z) \cap I_{00}^*(z^*)\} \\ \cup \{e_{n+q+m+k} | k \in I_\Phi(z) \cap I_{+0}^*(z^*)\} \\ \cup \{e_{n+q+m+k} | k \in I_\Phi(z) \cap I_{00}^*(z^*)\} \end{array} \tag{V.2}$$

are positively linearly independent.

Inspired by Lemma 5.2, we can multiply any terms by some positive number and the new family vectors will remains positively linearly independent.

$$\begin{array}{l} \{\nabla F_j | j = 1, \dots p+q+2m\} \cup \{-e_{n+i} | i \in I_0(z)\} \\ \cup \{e_{n+q+k} | k \in I_1(z)\} \cup \{e_{n+q+m+l} | l \in I_2(z)\} \\ \cup \{\partial_{u_r}\theta(t_k, u_r) e_{n+q+r} | r \in I_\Phi(z) \cap I_{0+}^*(z^*)\} \\ \cup \{\max(\partial_{u_r}\theta(t, u_r), \partial_{v_r}\theta(t, v_r)) \, e_{n+q+r} | r \in I_\Phi(z) \cap I_{00}^*(z^*)\} \\ \cup \{\partial_{v_r}\theta(t, v_r) e_{n+q+r} | r \in I_\Phi(z) \cap I_{+0}^*(z^*)\} \\ \cup \{\partial_{v_k}\theta(t, v_k) e_{n+q+m+k} | k \in I_\Phi(z) \cap I_{+0}^*(z^*)\} \\ \cup \{\max(\partial_{u_r}\theta(t, u_r), \partial_{v_r}\theta(t, v_r)) \, e_{n+q+m+k} | k \in I_\Phi(z) \cap I_{00}^*(z^*)\} \\ \cup \{\partial_{u_k}\theta(t, u_k) e_{n+q+m+k} | k \in I_\Phi(z) \cap I_{0+}^*(z^*)\} \end{array} \tag{V.3}$$

are positively linearly independent.

Now we have to check that MFCQ holds for the problem (2_t) for $z \in \mathcal{U} \cap \mathcal{Z}_t$, with $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ and $\bar{t} = \min\{\bar{t}_1, \bar{t}_2\}$. We should to show that :

$$\begin{array}{l} \sum_{j=1}^{p+q+2m} \lambda_j \nabla F_j - \sum_{i \in I_0(z)} \nu_i e_{n+i} + \sum_{k \in I_1(z)} \mu_{1,k} e_{n+q+k} \\ + \sum_{l \in I_2(z)} \mu_{2,l} e_{n+q+m+l} + \sum_{r \in I_\Phi(z)} \eta_r^\Phi \partial_{u_r}\theta(t, u_r) e_{n+q+r} \\ + \sum_{r \in I_\Phi(z)} \eta_r^\Phi \partial_{v_r}\theta(t, v_r) e_{n+q+m+r} = 0. \end{array} \tag{V.4}$$

with $\lambda_j \geq 0, \nu_i \geq 0, \mu_{1,k} \geq 0, \mu_{2,l} \geq 0, \eta_r \geq 0$, holds only for the trivial solution ($\lambda_j = \nu_i = \mu_{1,k} = \mu_{2,l} = \eta_r^\Phi = 0$).

By spliting the index set I_Φ: $I_\Phi(z) = (I_\Phi(z) \cap I_{+0}^*(z^*)) \cup (I_\Phi(z) \cap I_{00}^*(z^*)) \cup (I_\Phi(z) \cap I_{0+}^*(z^*))$. From (V.3) it follows that:

$$\begin{cases} \lambda_j = 0, \nu_i = 0, \mu_{1,k} = 0, \mu_{2,l} = 0, \\ \eta_r^\Phi \partial_{u_r}\theta(t, u_r) = 0 \quad \text{for } r \in I_\Phi(z) \cap I_{0+}^*(z^*) \\ \eta_r^\Phi \max(\partial_{u_r}\theta(t, u_r), \partial_{v_r}\theta(t, v_r)) = 0 \quad \text{for } r \in I_\Phi(z) \cap I_{00}^*(z^*) \\ \eta_r^\Phi \partial_{v_r}\theta(t, v_r) = 0 \quad \text{for } r \in I_\Phi(z) \cap I_{+0}^*(z^*) \end{cases}$$

By lemma 5.2, we obtain that $\eta_r = 0$ so the standard MFCQ are satisfied for the problem (2_t) .

The following theorem identifies a relation between the optimal solutions of the original MPCC problem and the penalized-regularized problem under the MPCC-MFCQ assumption.

Denote, for k be sufficiently large $I_S := \{i/u_i^* > 0, v_i^* > 0\}$.

Theorem 5.4: Let $\{\sigma_k\}$ be a non-decreasing sequence converging to $+\infty$. Let (z^k, t_k) be a strong ϵ_k-stationary point of the penalized-regularized problem with $\epsilon_k = o(t_k)$. If (z^*, t_*) is a cluster point of $\{(z^k, t_k)\}$, z^* is feasible for $\widetilde{(1)}$ and the MPCC-MFCQ holds at z^*, then:

i) $t_* = 0$, and
ii) If $I_S \cap supp(\eta^*) = \emptyset$ then z^* is an M-stationary point.
If $I_S \cap supp(\eta^*) \neq \emptyset$ then z^* is an C-stationary point.

i) The first line of the definition of the strong $\epsilon-$ stationary is as follows:

$$\|\nabla_x f^k + t_k^{-1} \sum_{j=1}^{p+q+2m} (F_j^k - t_k w) \nabla_x F_j^k\|_\infty \leq \epsilon_k$$

$$\begin{array}{c} \left| - t_k^{-1}(F_i^k - t_k w_i) - \nu_i^k \right| \leq \epsilon_k, \quad i = 1, \dots, q \\ \left| - t_k^{-1}(F_{p+q+l}^k - t_k w_l) - \hat{\mu}_{1,l}^k \right| \leq \epsilon_k, \quad l = 1, \dots, m \\ \left| - t_k^{-1}(F_{p+q+m+l}^k - t_k w_l) - \hat{\mu}_{2,l}^k \right| \leq \epsilon_k, \quad l = 1, \dots, m \end{array} \tag{V.5}$$

with

$$\begin{array}{rcl} F_i^k & = & F_i(z^k) \\ \widetilde{\mu}_{1,l}^k & = & \mu_{1,l}^k - \eta_l^k \partial_{u_l} \theta(t_k, u_l^k) \\ \widetilde{\mu}_{2,l}^k & = & \mu_{2,l}^k - \eta_l^k \partial_{v_l} \theta(t_k, v_l^k) \end{array}$$

We will prove by contradiction that if $t_* \neq 0, \sigma_k$ can not converge to $+\infty$. We suppose there exist a subsequence (z^k, t_k) that converges the $\{(z^*, t_*)\}$ such that $t_* \neq 0$. Let $\pi^k = -t_k^{-1}(F(z^k) - t_k w)$, this sequence is bounded because F_j are continuously differentiable, so the sequence $\{(\nu_i^k, \widetilde{\mu}_{1,l}^k, \widetilde{\mu}_{2,l}^k)\}$ is bounded.

By the second line of the definition of the strong $\epsilon-$ stationary (Definition.3)

$$\Big| - \tfrac{1}{2} t_k^{-2} \Delta_k - t_k^{-1} \sum_{j=1}^{p+q+2m} (F_j^k - t_k w) + \sum_{i=1}^{m} (\mu_{2,i}^k + \eta_i^k (\partial_t \theta(t_k, u_i^k) + \partial_t \theta(t_k, v_i^k))) + \sigma_k \frac{1}{2\sqrt{t_k}} \Big| \leq \epsilon_k.$$

By multiplying by $\sqrt{t_k}$ we obtain

$$\Big| - \tfrac{1}{2} t_k^{-3/2} \Delta_k - t_k^{-1/2} \sum_{j=1}^{p+q+2m} (F_j^k - t_k w) + t_k^{1/2} \sum_{i=1}^{m} (\mu_{2,i}^k + \eta_i^k (\partial_t \theta(t_k, u_i^k) + \partial_t \theta(t_k, v_i^k))) + \frac{\sigma_k}{2} \Big| \leq \epsilon_k \sqrt{t_k} \tag{V.6}$$

In this inequality the first and second terms of the left side are both bounded.
We have to show that the third term is bounded. This term can be written as

$$\sum_{i=1}^{m} \left(\mu_{2,i}^k + \eta_i^k (\partial_t \theta(t_k, u_i^k) + \partial_t \theta(t_k, v_i^k)) \right) = \sum_{i \in I_\Phi(z^k)} \left(\mu_{2,i}^k + \eta_i^k (\partial_t \theta(t_k, u_i^k) + \partial_t \theta(t_k, v_i^k)) \right) + \sum_{i \notin I_\Phi(z^k)} \left(\mu_{2,i}^k + \eta_i^k (\partial_t \theta(t_k, u_i^k) + \partial_t \theta(t_k, v_i^k)) \right)$$

We have to prove that these two sums are bounded :

- For $l \notin I_\Phi(z^k)$ then $\eta_l^k = 0$ (by the definition of the strong $\epsilon-$ stationary). We have $\widetilde{\mu}_{2,l}^k = \mu_{2,l}^k - \eta_l^k \partial_{v_r} \theta(t_k, v_r^k)$, so $\widetilde{\mu}_{2,l}^k = \mu_{2,l}^k$ and since $\widetilde{\mu}_{2,l}^k$ is bounded then $\mu_{2,l}^k$ is bounded .

- For $l \in I_\Phi(z^k)$ then $\theta(t_k, u_l^k) + \theta(t_k, v_l^k) - 1 = 0$. By lemma 5.1, we have $I_1(z^k) \cap I_\Phi(z^k) = \emptyset$ and $I_2(z^k) \cap I_\Phi(z^k) = \emptyset$, so, $l \notin I_1(z^k)(u_l^k > 0)$ and $l \notin I_2(z^k)(v_l^k > -t_k)$.
By the definition of the strong $\epsilon-$ stationary), we have :
$$-\epsilon_k \leq \mu_{2,l}^k (v_l^k + t_k) \leq \epsilon_k$$
since t_k tends to $t_* \neq 0$, there exists $b > 0$ such that $t_k \geq b$ for all k.
$$0 \leq \mu_{2,l}^k \leq \frac{\epsilon_k}{t_k} \leq \frac{\epsilon_k}{b}.$$
Then, $\mu_{2,l}^k$ is bounded.
Now, let us to show that η_l^k is bounded.
We have
$$\widetilde{\mu}_{2,l}^k = \mu_{2,l}^k - \eta_l^k \partial_{v_l} \theta(t_k, v_l^k),$$
then $|\eta_l^k| = \frac{|\mu_{2,l}^k - \widetilde{\mu}_{2,l}^k|}{\partial_{v_l} \theta(t_k, v_l^k)}$. $\widetilde{\mu}_{2,l}^k$ and $\mu_{2,l}^k$ are bounded, it remains to show that $\partial_{v_l} \theta(t_k, v_l^k)$ is bounded below. Since the sequence $v_l^k \longrightarrow v_l^*, v_l^k$ is bounded and there exist $\lambda > 0$ such that $v_l^k < \lambda$. In other hand, $t_k \longrightarrow t_* \neq 0$ then there exist $b > 0$ such that $t_k > b \neq 0$. By the concavity of θ by respect to the second variable:
$$\partial_{v_l} \theta(t_k, v_l^k) \geq \partial_{v_l} \theta(t_k, \lambda) \geq \partial_{v_l} \theta(b, \lambda) > 0.$$

So, $\partial_{v_l}\theta(t_k, v_l^k)$ is bounded below and we have the result.

Since $\sigma_k \to \infty$, we obtain a contradiction in (V.6), so, we can conclude that $t_* = 0$.

ii)We can write (V.5) as:

$$\begin{aligned}&\|\nabla_x f^k + \sum_{i=1}^{q}[(\nu_i^k - \Pi_i^k) - \nu_i^k]\nabla_x F_i^k \\ &+ \sum_{l=1}^{m}[(\hat{\mu}_{1,l}^k - \Pi_{p+q+l}^k) - \hat{\mu}_{1,l}^k]\nabla_x F_{p+q+l}^k \\ &+ \sum_{l=1}^{m}[(\hat{\mu}_{2,l}^k - \Pi_{p+q+m+l}^k) - \hat{\mu}_{2,l}^k]\nabla_x F_{p+q+m+l}^k \\ &- \sum_{j=1}^{p}\Pi_{q+j}^k)\nabla_x F_{p+j}^k\|_\infty \leq \epsilon_k \\ &\left|\nu_i^k - \Pi_i^k\right| \leq \epsilon_k, \quad i = 1,\ldots,q \\ &\left|\hat{\mu}_{1,l}^k - \Pi_{p+q+l}^k - \right| \leq \epsilon_k, \quad l = 1,\ldots,m \\ &\left|\hat{\mu}_{2,l}^k - \Pi_{p+q+m+l}^k\right| \leq \epsilon_k, \quad l = 1,\ldots,m\end{aligned} \tag{V.7}$$

Let's show that $\{(\nu^k, \beta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)\}$ is bounded, where β^k denotes the subsequence restricted to coefficients of equality constraints.
Assume that the sequence $\{(\nu^k, \beta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)\}$ is unbounded, then we can find a subsequence such that

$$\frac{(\nu^k, \beta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)}{\|\ (\nu^k, \beta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)\ \|} \longrightarrow (\overline{\nu}, \overline{\beta}, \overline{\mu}_1, \overline{\mu}_2) \neq 0$$

Using the inequalities (V.7), and since the gradient vectors of F are bounded (F is continuously differentiable), and dividing by the term $\|\ (\nu^k, \beta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)\ \|$, we obtain at the limit:

$$0 = \sum_{i=1}^{q}\overline{\nu}_i\nabla_x F_i^* + \sum_{j=1}^{p}\overline{\beta}_j\nabla_x F_{j+q}^* + \sum_{l=1}^{m}\overline{\mu}_{1,l}\nabla_x F_{p+q+l}^* + \sum_{l=1}^{m}\overline{\mu}_{2,l}\nabla_x F_{p+q+m+l}^*$$

with $\nabla_x F_i^* = \nabla_x F_i(z^*)$.
Now, we have to show that $\overline{\nu} \geq 0, \overline{\nu}_i = 0, i \notin I_0^*(z^*)$, $\overline{\mu}_{1,l} = 0, l \notin I_{00}^*(z^*) \cup I_{0+}^*(z^*)$ and $\overline{\mu}_{1,l} = 0, l \notin I_{00}^*(z^*) \cup I_{+0}^*(z^*)$.

The definition of the strong $\epsilon-$ stationary we have $\overline{\nu} \geq 0$. Suppose that $\overline{\nu}_i^k > 0$, so there exist some constant $c > 0$ and all k sufficiently large. This yields $|s_i^k| \leq \frac{\epsilon_k}{c} \longrightarrow 0$, so $s_i^* = 0$.
Let $l \in \{1,\ldots,m\}$ such that $u_l^* > 0$ and suppose that $\overline{\mu}_{1,l} \neq 0$. We have $u_l^k \longmapsto u_l^*$ and $|\hat{\mu}_{1,l}^k| > c$ for k sufficiently large. By the definition of the strong $\epsilon-$ stationary we have $0 \leq \mu_{1,l}^k \leq \frac{\epsilon_k}{u_l^k} \longrightarrow 0$.
But, $\widetilde{\mu}_{1,l}^k = \mu_{1,l}^k - \eta_l^k\partial_{u_l}\theta(t_k, u_l^k) \longrightarrow 0$ (for $u_l^* > 0$ we have $\partial_{u_l}\theta(t_k, u_l^k) \to 0$), so we obtain a contradiction. We deduce that $\overline{\mu}_{1,l} = 0$, same proof for $\overline{\mu}_{2,l}$.

So, we have the following positive linear combination:

$$\begin{aligned}&\sum_{i\in I_0^*(z^*)}\overline{\nu}_i\nabla_z F_i^* + \sum_{j=1}^{p}\overline{\theta}_j\nabla_z F_{j+q}^* \\ &+ \sum_{I_{00}^*(z^*)\cup I_{0+}^*(z^*)}\overline{\mu}_{1,l}\nabla_z F_{p+q+l}^* \\ &+ \sum_{I_{00}^*(z^*)\cup I_{+0}^*(z^*)}\overline{\mu}_{2,l}\nabla_z F_{p+q+m+l}^* + \sum_{i\in I_0^*(z^*)}\overline{\nu}_i e_{p+i} \\ &+ \sum_{k\in I_{00}^*(z^*)\cup I_{0+}^*(z^*)}\overline{\mu}_{1,l}e_{p+q+k} \\ &+ \sum_{l\in I_{00}^*(z^*)\cup I_{+0}^*(z^*)}\overline{\mu}_{2,l}e_{p+q+m+l} = 0\end{aligned}$$

By the given that MPCC-MFCQ is satisfied at z^* and the previous theorem, implies $(\overline{\nu}_i, \overline{\theta}_j, \overline{\mu}_{1,l}, \overline{\mu}_{2,l}) = 0$, contradiction.
Consequently the sequence $(\nu^k, \theta^k, \hat{\mu}_1^k, \hat{\mu}_2^k)$ is bounded and so there exist a subsequence that converges to some limit point $(\nu^*, \theta^*, \hat{\mu}_1^*, \hat{\mu}_2^*)$, which satisfies the following conditions:

$$\begin{aligned}&\nabla_x f^* - (\nu^*, \theta^*, \hat{\mu}_1^*, \hat{\mu}_2^*)^T\nabla_x F^* = 0 \\ &\nu^* \geq 0, supp(\nu^*) \subseteq I_0^*, \\ &supp(\mu_1^*) \subseteq I_{00}^*(z^*) \cup I_{0+}^*(z^*), \\ &supp(\mu_2^*) \subseteq I_{00}^*(z^*) \cup I_{+0}^*(z^*).\end{aligned}$$

So, z^* is W-stationary point.

Suppose now that there exist some indexes l such that $u_l^* = v_l^* = 0$. By taking a sub-sequence of (z^k, t_k) strong $\epsilon-$ stationary point, in these situations, we can only meet the following situations
Case 1: For every k assigned $u_l^k = 0, v_l^k \geq 0 > -t_k$.
By the definition of the strong $\epsilon-$ stationary

$$0 \leq \mu_{2,l}^k(v_l^k + t_k) \leq \epsilon_k$$

then

$$0 \leq \mu_{2,l}^k \leq \frac{\epsilon_k}{v_l^k + t_k} \leq \frac{\epsilon_k}{t_k}$$

when $k \to \infty$, $\mu_{2,l}^k \to 0$ since $\epsilon_k = o(t_k)$.
On the other hand, we have $\theta(t_k, u_l^k) + \theta(t_k, v_l^k) - 1 \neq 0$, then $\eta_l^k = 0$.
$\widetilde{\mu}_{2,l}^k = \mu_{2,l}^k - \eta_l^k\partial_{v_l}\theta(t, v_l^k) \to \widetilde{\mu}_{2,l}^* = 0$.

Case 2: $u_l^k = 0, (v_l^k \geq -t_k$ and $v_l^k < 0)$.
In this case, we have $\theta(t_k, u_l^k) + \theta(t_k, v_l^k) - 1 \neq 0$, so

$\eta_l^k = 0$.
Then, $\widetilde{\mu}_{1,l}^k = \mu_{1,l}^k \geq 0$ and $\widetilde{\mu}_{2,l}^k = \mu_{2,l}^k \geq 0$ so $\widetilde{\mu}_{1,l}^* \geq 0, \widetilde{\mu}_{2,l}^* \geq 0$.

Case 3: $u_l^k > 0, v_l^k < 0, v_l^k \geq -t_k$.
Same proof as case 2.

Case 4: $u_l^k > 0, v_l^k \geq 0$, and $(\theta(t_k, u_l^k) + \theta(t_k, v_l^k) - 1 \neq 0)$:
Same proof as case 1.
Case 5: $u_l^k > 0, v_l^k > 0$ and $(\theta(t_k, u_l^k) + \theta(t_k, v_l^k) - 1 = 0)$. By the definition of the strong $\epsilon-$ stationary

$$0 \leq \mu_{1,l}^k u_l^k \leq \epsilon_k$$

then

$$\mu_{1,l}^k \leq \frac{\epsilon_k}{t_k}$$

when $k \to \infty$, $\mu_{1,l}^k \to 0$ since $\epsilon_k = o(t_k)$.
Same, by the definition of the strong $\epsilon-$ stationary,

$$0 \leq \mu_{2,l}^k \leq \frac{\epsilon_k}{v_l^k + t_k} \leq \frac{\epsilon_k}{t_k}$$

when $k \to \infty$, $\mu_{2,l}^k \to 0$ since $\epsilon_k = o(t_k)$.
So, $\widetilde{\mu}_{1,l}^* \widetilde{\mu}_{2,l}^* = (\eta_l^*)^2 \partial_{u_l} \theta(t_k, u_l^*) \partial_{v_l} \theta(t_k, u_l^*)$.
If $I_S \cap \eta_l^* \neq \emptyset$, we can meet the case 5 and we obtain a C-stationary point.
If $I_S \cap \eta_l^* = \emptyset$ then we obtain a M-stationary point.

VI. Numerical results

In this section, we present some preliminary numerical results. These simulations have been done using AMPL language [6], with the the KNITRO solver. Our aim is just to verify the qualitative numerical efficiency of our approach. Our algorithm to solve the original MPCC is :

We use a subset of the MacMPEC [19] test problems

Algorithm

1. Choose a starting point (z^0, t_0) and σ_0 . Set $k = 0$.

2. While the stopping criterion for the MPCCs is not satisfied (i.e. $\Delta(z^k; t_k) > tol$) do
(Δ is defined in (IV.2)).
Find an approximate solution (z^{k+1}, t_{k+1}) of the penalized-relaxed problem (P_σ) , by using (z^k, t_k) as a starting point.
Let $\sigma_{k+1} = \alpha \sigma_k$.

with known optimal values and solutions. In all our tests, we use the same function β defined by $\beta(t) := \sqrt{t}$ as in [16]. The starting point is almost the default value given by the solver and does not influence the solution. The obtained solution is obtained when standard approximate stationary and the feasibility conditions are satisfied. The feasibility is reached if the constraints violation measure and the variable t are equal to zero up to the prescribed tolerance.
The following tables give for each considered problem and for different starting points, the final value of the variable t_*, the objective value (Obj.val) of the MPCC, the constraint violation Δ_*, and the final value of the penalty parameter σ_*. In our experiments, we used $tol = 1e-5$ and we made a logarithmic scaling for our two functions to bound their gradients.
Each constraint

$$\theta(u_i, t) + \theta(v_i, t) \leq 1$$

in the case of the θ_t^1 function

$$1 - \frac{t}{u_i + t} + 1 - \frac{t}{v_i + t} \geq 1$$

is in fact replaced by the following inequality

$$\ln\left(\frac{t}{u_i + t} + \frac{t}{v_i + t}\right) \geq 0,$$

in the case of the θ_t^1 function and

$$t \ln\left(e^{-\frac{u_i}{t}} + e^{-\frac{v_i}{t}}\right) \geq 0.$$

and in the case of the θ_t^2 function.

TABLE I
USING θ^1-FUNCTION

Problem	t_*	Start	Obj.val	Δ^*	σ_*
Bard1A	$1.17e-05$	n/a	16.9999	$2.56e-09$	8
bilevel1	$1e-09$	n/a	-0.00063	$9.88e-08$	16
bilevel2	$5e-11$	n/a	-6600	$1.8e-11$	32768
bilevel3	$1e-06$	n/a	12.6787	$1.09e-11$	2
dempe	$1e-07$	n/a	28.25	$4.90e-13$	16
desilva	$1e-07$	n/a	-0.99999	$3.10e-12$	2
df1	$1e-09$	n/a	$1.87e-13$	$3.29e-22$	2
Gauvin	$1.4e-05$	n/a	20	$6.21e-9$	4
hs044-i	$4e-06$	(1,1,1,1)	$7.46e-06$	$1e-19$	2
jr1	$1e-08$	n/a	0.5	$1e-16$	2
scholtes1	$1e-07$	n/a	2.0	$9.57e-14$	2
nash1	$7e-05$	nash1a.data	$1.7e-08$	$2.05e-09$	2
	0.0002	nash1b.data	$7.35e-06$	$1.14e-08$	2
	0.0001	nash1c.data	$3.99e-06$	$1.61e-09$	2
	$4e-06$	nash1d.data	$3.13e-06$	$1.56e-10$	2
	0.0001	nash1e.data	$7.3e-07$	$3.73e-09$	2
gnash1	$2e-07$	gnash10.data	-230.82	$2.54e-11$	32768
	$1e-05$	gnash11.data	-129.91	$3.33e-08$	256
	$1e-06$	gnash12.data	-36.93	$6.1e-07$	32768
stackelberg1	$1e-10$	n/a	-3266.67	$9.58e-15$	512

VII. Conclusion

We proposed a penalization and a relaxation scheme to solve the MPCC problems. Under the MPCC-Mangasarian-Fromovitz constraints qualifications, we showed the link between the penalized problem and the MPCC, by proving that any accumulation point of the sequence of strong approximate first-order points

TABLE II
USING θ^2-FUNCTION

Problem	t_*	Start	Obj.val	Δ^*	σ_*
Bard1A	$1e-08$	n/a	17	$6e-13$	2
bilevel1	$1.2e-06$	n/a	$1.7e-05$	$5.17e-10$	4
bilevel2	$1e-10$	n/a	-6600	$1.37e-12$	32768
bilevel3	$1e-06$	n/a	-7.32144	$5.5e-07$	$9e+15$
dempe	$1e-07$	n/a	28.25	$4.90e-13$	16
desilva	$2e-06$	n/a	-0.9999	$3.9e-12$	2
df1	$1e-09$	n/a	$1.2e-09$	$1.2e-18$	2
Gauvin	$2.68e-08$	n/a	20	$2e-13$	4
jr1	$1e-08$	n/a	0.5	$1e-16$	2
scholtes1	$1e-09$	n/a	2.25	$2.9e-18$	2
nash1	$1e-07$	nash1a.data	$5.4e-11$	$2.5e-11$	2
	$4e-08$	nash1b.data	$3.4e-11$	$1e-12$	2
	$2.6e-07$	nash1c.data	0.5	$1e-11$	2
	$4.3e-08$	nash1d.data	0.5	$6.6e-12$	2
	$5e-08$	nash1e.data	$4.6e-09$	$5.8e-12$	2
hs044-i	$1.7e-05$	$(1,1,1,1)$	$7.2e-06$	$2.25e-09$	2
gnash1	$6.5e-08$	gnash10.data	-230.823	$2e-07$	32768
	$1e-05$	gnash11.data	-129.915	$2.9e-08$	256
	$1e-06$	gnash12.data	-36.9331	$1.8e-10$	64
stackelberg1	$1e-10$	n/a	-3266.67	$5.48e-15$	256

generated by our scheme is M-stationary for MPCC. To illustrate the effectiveness of our approach, we tested our method on several examples from the MacMpec library and obtained very promising results; This a first step and we plan in future work, to consider realistic and large-scale problems.

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