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Portfolio Optimization under Non-Normal Returns: Why Higher-Order Moments Matter in the Mean–CVaR Model

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***Abstract*—importance of considering higher order moments in the estimation of the CVaR, when the returns are not normally distributed. The study focuses on a portfolio of 10 stocks listed on the New York Stock Exchange. The results of the portfolio's optimization by the models mean-parametric CVaR and mean-Cornish Fisher CVaR, showed that the risk estimated by Cornish Fisher CVaR, which considers the Skewness and the Kurtosis, is higher than that evaluated by parametric CVaR, which is based only on the mean and the standard deviation in the estimation of the risk.**

***Index Terms*—Portfolio, Optimization, CVaR, Skewness, Kurtosis.**

I. Introduction

Portfolio optimization has always been a concern for researchers in financial mathematics. In this context, Harry Markowitz (1952) introduced the "mean-variance" model. This model uses the variance of portfolio returns as a measure of risk. In effect, the Markowitz model consists of minimizing variance (or standard deviation) for a given level of return or maximizing return for a given level of risk. Advances and research in the field have allowed the use of more mature indicators, such as Value at Risk (VaR) or Conditional Value at Risk (CVaR). VaR can be defined as the worst loss expected from a portfolio, for a certain confidence level (1-), over a time horizon t, assuming an adverse market scenario.

Although VaR is a popular and intuitive risk measure, it has two important limitations. First, it does not satisfy the sub-additivity property. Second, it provides no information about the magnitude of losses exceeding the VaR at level . An alternative risk measure is the Conditional Value at Risk (CVaR), which avoids both limitations. CVaR is the expected loss given that losses exceed VaR.

Based on the Markowitz model, the mean-VaR and mean-CVaR models were developed. However, these models are based on the parametric estimation of VaR and CVaR, which assumes that asset returns are normally distributed. This assumption is problematic because financial returns are often not normal. Moreover, the mean-CVaR model assumes a quadratic utility function, in the sense that investors are only interested in the first two moments (mean and variance), while in reality, preferences may extend to higher-order moments such as skewness and kurtosis.

Tobin (1958) and Rubinstein (1973) have already shown that, in choosing a portfolio, higher order moments must be considered, and that the investor's utility function is not quadratic. Arditti (1975) and Krauss and Litzenberger (1976) have shown that the investor generally seeks a strictly positive skewness, and thus the investor's utility function cannot be quadratic. Scott and Horvath (1980) showed that investors have a positive preference for odd central moments and a negative preference for even central moments. Chunhachinda et al. (1997) analyzed in detail the preference of investors for skewness.

Indeed, a negative skewness means that large negative returns occur more frequently than large positive returns. Conversely, if the skewness coefficient is positive, it indicates that large positive returns occur more frequently than large negative returns. On the other hand, kurtosis reflects the presence of extreme events: if kurtosis exceeds 3, the distribution is said to be leptokurtic; otherwise, it is said to be platykurtic.

Empirically, the normality assumption is not satisfied, which has pushed several authors to seek solutions to this problem. Yao et al. (2013) obtained an estimated calculation formula of CVaR using nonparametric loss function density estimation and formulated two nonparametric mean-CVaR portfolio selection models based on two methods of bandwidth selection. Jaman et al. (2014) replaced variance with CVaR in the mean-variance-skewness-kurtosis model and showed that CVaR is a better measure of risk than variance in portfolio optimization. Zhao et al. (2015) proposed a mean-CVaR-skewness portfolio optimization model based on the asymmetric Laplace distribution, which is appropriate to describe the leptokurtosis and asymmetry of financial assets. They added skewness to the model and proposed a simplified model to solve the multi-objective optimization problem. Zhai et al. (2018) designed a hybrid smart algorithm to solve the mean-risk-skewness model they proposed.

In our study we will focus on the Cornish-Fisher expansion, as a solution to the case of non-normality of returns, which is commonly used by academics and practitioners in portfolio allocation and risk management. It is a simple tool to determine the quantiles of non-normal distributions. It allows us to consider skewness and kurtosis, through a formula in which higher-order moment terms are explicitly included (Amédée-Manesme et al., 2019).

The current challenge is as follows: Does ignoring higher-order moments in mean-CVaR optimization lead to a significant underestimation of portfolio risk, and how important is it to incorporate skewness and kurtosis for more accurate risk assessment?

The following hypotheses are proposed based on the literature review:

- H0: The parametric CVaR underestimates the risk, and the consideration of higher-order moments is important in the estimation of CVaR.
- H1: The parametric CVaR does not underestimate the risk, and the consideration of higher-order moments is not important in the estimation of CVaR.

II. Methodology & Data

The methodology of this study is based on the optimization of a portfolio composed of a set of stocks listed on the New York Stock Exchange, first using the mean-parametric CVaR model, and secondly using the mean-Cornish-Fisher CVaR model, in order to compare the results obtained.

A. Portfolio Stocks Sample

We have chosen to build this portfolio from 10 stocks, listed on the New York Stock Exchange, from different sectors to have a well-diversified portfolio.

TABLE I
Portfolio stocks sample

Companies	Ticker symbol	Sectors
ExxonMobil	NYSE:XOM	Energy
DuPont	NYSE:DD	Materials
Boeing	NYSE:BA	Industrials
Nike	NYSE:NKE	Consumer Discretionary
Coca-Cola	NYSE:KO	Consumer Staples
Pfizer	NYSE:PFE	Health Care
JPMorgan Chase	NYSE:JPM	Financials
IBM	NYSE:IBM	Information Technology
Verizon Communications	NYSE:VZ	Communication Services
Southern Company	NYSE:SO	Utilities

Source: Author's own elaboration

B. Data

The analysis is based on a series of data concerning daily stock prices during the period from 03/01/2012 to 30/12/2022 for each company.

We obtain the return at time t for each stock using the formula:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{II.1}$$

Where P_t : the stock's value on the market at t;
And P_{t-1}: the stock's value on the market at $t-1$.

C. CVaR in Portfolio Optimization

The parametric CVaR and Cornish-Fisher CVaR optimization programs will be presented in this section.

1) Parametric CVaR Optimization Program: The parametric method assumes that equity returns follow a normal distribution, and that the $CVaR_\alpha$ of a portfolio P, at a confidence level of $1-\alpha$, can be estimated using the portfolio's mean return μ_p and standard deviation σ_p. According to Huang (2000), the $CVaR_\alpha$ equation can be written as the following density formula:

$$CVaR_\alpha = \mu_p + Z_\alpha \sigma_p \tag{II.2}$$

Where:

$$Z_\alpha = \frac{-e^{-1/2z_\alpha^2}}{\alpha\sqrt{2\pi}} \tag{II.3}$$

z_α : the order α quantile of the standard normal distribution

Thus, the CVaR optimization program for a portfolio is written as follows:

$$\min_X \left| X^T\mu + Z_\alpha\sqrt{X^T M X} \right| \tag{II.4}$$

subject to:

$$X^T\mu = E^* \tag{II.5}$$

$$X^T\mathbf{1} = 1 \tag{II.6}$$

$$X_i \geq 0 \tag{II.7}$$

Where:
μ : expected returns vector of stocks;
X : stock proportions vector in portfolio;
$\mathbf{1} = (1,\ldots,1)$: unit vector of N dimension;
M : variance-covariance matrix;
E^* : (constant) given expected return of the portfolio.

2) Cornish-Fisher CVaR Optimization Program: When asset returns do not follow a normal distribution, the parametric CVaR would underestimate risk. Thus, adjustments for skewness and kurtosis were implemented in the variance-covariance model of CVaR. This approximation, based on the Taylor series, uses the moments of the distribution that deviate from the normal to calculate its percentiles. According to Maillard (2012), the Cornish-Fisher expansion is used to fit the Z_α of the traditional CVaR, when asset returns do not follow a normal distribution, such that:

$$W_\alpha = \frac{-e^{-1/2z_\alpha^2}}{\alpha\sqrt{2\pi}}\left[1 + Z_\alpha\left(\frac{S}{6}\right) + (1-2Z_\alpha^2)\frac{S^2}{36} + (-1+Z_\alpha^2)\left(\frac{K}{24}\right)\right] \quad \text{(II.8)}$$

Hence, the Cornish-Fisher CVaR optimization program for a portfolio is written as follows:

$$\min_X \left| X^T\mu + W_\alpha\sqrt{X^T M X}\right| \quad \text{(II.9)}$$

subject to:

$$X^T\mu = E^* \quad \text{(II.10)}$$
$$X^T\mathbf{1} = 1 \quad \text{(II.11)}$$
$$X_i \geq 0 \quad \text{(II.12)}$$

III. Results

Before presenting the results of the portfolio optimization, it is important to verify stocks returns normality.

A. Verification of the normality of returns

The results of the Jarque-bera normality test are presented as follows in Table 2.

TABLE II
NORMALITY TEST OF STOCKS RETURNS

Companies	Skewness	Kurtosis	Test Value	Proba
NYSE:XOM	-0.165409	11.01714	7422.934	0.0000
NYSE:DD	0.103624	10.45335	6409.691	0.0000
NYSE:BA	-0.543452	24.44079	53136.63	0.0000
NYSE:NKE	0.287194	14.03466	14076.38	0.0000
NYSE:KO	-0.858887	13.41912	12856.03	0.0000
NYSE:PFE	0.153746	8.711631	3772.029	0.0000
NYSE:JPM	-0.101892	15.65106	18457.13	0.0000
NYSE:IBM	-0.789558	13.73453	13572.57	0.0000
NYSE:VZ	-0.066413	7.447287	2282.314	0.0000
NYSE:SO	0.203200	28.68250	76064.34	0.0000

According to the Jarque–Bera test, all stock return distributions are non-normal, particularly since their kurtosis values are much higher than the normal benchmark of 3, indicating that they are leptokurtic.

B. Results of the Portfolio Optimization

The study results are summarized in the following tables: Table 3 reports the portfolios optimized using the mean–parametric CVaR model, while Table 4 reports those optimized using the mean–Cornish Fisher CVaR model.

According to Tables 3 and 4 below, the results of the portfolio optimization reveal distinct allocation patterns under the two models.

According to Table 3, all portfolios do not contain NYSE:DD and NYSE:BA after optimization using the mean-parametric CVaR model. In portfolio 1, corresponding to the minimal CVaR (0.018435414), NYSE:KO, NYSE:PFE, and NYSE:VZ have the highest weights, representing more than 70% of the portfolio. On the other hand, portfolio 9, associated with the highest mean return, consists only of NYSE:NKE.

According to Table 4, all portfolios do not contain NYSE:XOM, NYSE:DD, NYSE:BA, NYSE:IBM, and NYSE:SO after optimization using the mean-Cornish Fisher CVaR model. In portfolio 1, corresponding to the minimal CVaR (0.028681269), NYSE:PFE and NYSE:VZ dominate the allocation, accounting for more than 96% of the portfolio. In contrast, portfolio 9, associated with the highest mean return, consists exclusively of NYSE:NKE.

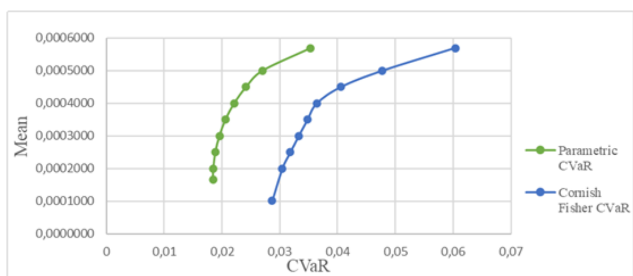

Fig. 1. Efficient Frontiers of Mean-Parametric CVaR and Mean-Cornish Fisher CvaR

Figure 1 shows that mean-Cornish Fisher CVaR efficient frontier is under mean-parametric CVaR efficient frontier.

Tables 3 and 4 and figure 1 prove that for a given mean return level, Cornish Fisher CVaR is higher than parametric CVaR. For example, portfolio 2 which has a mean of 0.0002, parametric CVaR (0.018501662) is lower than Cornish-Fisher CVaR (0.030428659).

TABLE III
PORTFOLIOS OPTIMIZED BY THE MEAN-PARAMETRIC CVAR MODEL

Portfolio	**1 (CVaR min)**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9 (Mean Max)**
Mean	0.0001664	0.0002000	0.0002500	0.0003000	0.0003500	0.0004000	0.0004500	0.0005000	0.0005695
St Dev	0.0090181	0.0090665	0.0092654	0.0096284	0.0101614	0.0109096	0.0119368	0.0133321	0.0174024
Skewness	-0.4391531	-0.4621051	-0.5085350	-0.5400704	-0.5255992	-0.4633764	-0.3705264	-0.2543904	0.2873498
Kurtosis	12.777005	12.741812	13.056199	13.592985	13.428956	11.527408	10.111049	10.051639	11.056799
α	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
z	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.644853627
Z	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.062712808
Parametric CVaR	0.018435414	0.018501662	0.018861885	0.019560688	0.020610012	0.022103441	0.024172179	0.027000226	0.035326583
Stock weight									
NYSE:XOM	0.037405475	0.028979195	0	0	0	0	0	0	0
NYSE:DD	0	0	0	0	0	0	0	0	0
NYSE:BA	0	0	0	0	0	0	0	0	0
NYSE:NKE	0.062258212	0.095200294	0.144226312	0.180334433	0.225421249	0.305232925	0.392574105	0.525654447	1
NYSE:KO	0.250770238	0.267560710	0.276049177	0.265651692	0.260418381	0.184189056	0.067877004	0	0
NYSE:PFE	0.177078497	0.203771064	0.236452721	0.258066357	0.286534889	0.304056398	0.313913709	0.185920366	0
NYSE:JPM	0	0.000176402	0.024101268	0.077302719	0.112799923	0.166421674	0.225635181	0.288425182	0
NYSE:IBM	0.042165190	0.007528701	0	0	0	0	0	0	0
NYSE:VZ	0.307761410	0.273097465	0.197099243	0.105453686	0.009743082	0	0	0	0
NYSE:SO	0.122560977	0.123686169	0.122071281	0.113191114	0.105082476	0.040099959	0	0	0

Source: Author's own elaboration

TABLE IV
PORTFOLIOS OPTIMIZED BY THE MEAN-CORNISH FISHER CVAR MODEL

Portfolio	**1 (CVaR min)**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9 (Mean Max)**
Mean	0.00010	0.00020	0.00025	0.00030	0.00035	0.00040	0.00045	0.00050	0.00057
St Dev	0.009979119	0.010126544	0.010296044	0.010697836	0.011303962	0.012024818	0.012493592	0.013517686	0.017402384
Skewness	-0.01189906	-0.07075662	-0.15731496	-0.20395444	-0.2090435	-0.1951683	-0.22510796	-0.14953065	0.287349798
Kurtosis	5.558918547	6.297584089	6.605835616	6.615974258	6.397023071	6.11158038	7.531875864	9.730223801	11.05679914
α	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
z	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.64485363	-1.644853627
Z	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.06271281	-2.062712808
W	-2.8842612	-3.02459169	-3.113734	-3.13733512	-3.10758663	-3.05931636	-3.28126066	-3.56792512	-3.500117889
CVaR-CF	0.028681269	0.030428659	0.031809142	0.033262696	0.03477804	0.036387724	0.040544733	0.047730091	0.060340858
Stock weight									
NYSE:XOM	0	0	0	0	0	0	0	0	0
NYSE:DD	0	0	0	0	0	0	0	0	0
NYSE:BA	0	0	0	0	0	0	0	0	0
NYSE:NKE	0	0.068811108	0.133399583	0.191957865	0.245742614	0.304841538	0.509859218	0.621894201	1
NYSE:KO	0.038783436	0	0	0	0	0	0	0	0
NYSE:PFE	0.291076944	0.497815256	0.537429637	0.587548339	0.645981746	0.695158466	0.490140774	0.236376891	0
NYSE:JPM	0	0	0	0	0	0	0	0.141728905	0
NYSE:IBM	0	0	0	0	0	0	0	0	0
NYSE:VZ	0.670139621	0.433373637	0.329170778	0.220493802	0.108275649	0	0	0	0
NYSE:SO	0	0	0	0	0	0	0	0	0

Source: Author's own elaboration

IV. DISCUSSION

The results of our study are consistent with the findings of Maillard (2012), who showed that parametric CVaR tends to underestimate risk when asset returns do not follow a normal distribution. In contrast, the Cornish–Fisher expansion overcomes this limitation, as it allows the adjustment of Z_α by incorporating skewness and kurtosis.

Our results are also consistent with the contributions of Tobin (1958) and Rubinstein (1973), who demonstrated that higher-order moments should be considered in portfolio selection. Furthermore, they align with the findings of Jaman et al. (2014), Zhao et al. (2015), and Zhai et al. (2018), who proposed optimization models based on CVaR estimation that account for higher-order moments and emphasized their importance.

V. Conclusion

In conclusion, when returns are not normally distributed, it is essential to consider skewness and kurtosis in risk estimation. The Cornish–Fisher expansion provides an effective framework to incorporate these higher-order moments into the estimation of CVaR.

Moreover, relying solely on parametric CVaR, which is based only on the mean and standard deviation, when returns do not follow a Gaussian distribution, may lead investors to underestimate the true level of risk.

References

[1] C. Amédée-Manesme, F. Barthélémy, and D. Maillard, "Computation of the corrected Cornish–Fisher expansion using the response surface methodology: application to VaR and CVaR," *Annals of Operations Research*, vol. 281, no. 1–2, pp. 423–453, 2019.

[2] F. D. Arditti, "Skewness and Investors' Decisions: A Reply," *Journal of Financial and Quantitative Analysis*, vol. 10, no. 1, pp. 173–176, 1975.

[3] P. Chunhachinda, K. Dandapani, S. Hamid, and A. Prakash, "Portfolio selection and skewness: Evidence from international stock markets," *Journal of Banking and Finance*, vol. 21, no. 2, pp. 143–167, 1997.

[4] H. M. Markowitz, "Portfolio selection," *Journal of Finance*, vol. 7, pp. 77–91, 1952.

[5] S. H. Jaaman, W. N. Lam, and Z. Isa, "A new higher moment portfolio optimisation model with conditional value at risk," *International Journal of Operational Research*, 2014.

[6] A. Kraus and R. H. Litzenberger, "Skewness preference and the valuation of risk assets," *Journal of Finance*, vol. 31, no. 4, pp. 1085–1100, 1976.

[7] D. Maillard, "A User's Guide to the Cornish Fisher Expansion," SSRN, 2012.

[8] M. P. Rubinstein, "A Mean-Variance Synthesis of Corporate Financial Theory," *Journal of Finance*, vol. 28, no. 1, pp. 167–181, 1973.

[9] R. A. Scott and P. A. Horvath, "On the Direction of Preference for Moments of Higher Order Than the Variance," *Journal of Finance*, vol. 35, no. 4, pp. 915–919, 1980.

[10] J. Tobin, "Liquidity Preference as Behavior Towards Risk," *The Review of Economic Studies*, vol. 25, no. 2, pp. 65, 1958.

[11] H. Yao, Z. Li, and Y. Lai, "Mean–CVaR portfolio selection: A nonparametric estimation framework," *Computers & Operations Research*, vol. 40, no. 4, pp. 1014–1022, 2013.

[12] J. Zhai, M. Bai, and H. Wu, "Mean-risk-skewness models for portfolio optimization based on uncertain measure," *Optimization*, vol. 67, no. 5, pp. 701–714, 2018.

[13] S. Zhao, Q. Lu, L. Han, Y. Liu, and F. Hu, "A mean-CVaR-skewness portfolio optimization model based on asymmetric Laplace distribution," *Annals of Operations Research*, vol. 226, no. 1, pp. 727–739, 2015.